343-754.
SR
1-22-74 XR 3,787,872

United States Patent [19]

Kauffman

[11] **3,787,872**

[45] **Jan. 22, 1974**

[54] **MICROWAVE LENS ANTENNA AND METHOD OF PRODUCING**

[75] Inventor: **James F. Kauffman**, Raleigh, N.C.

[73] Assignee: **Corning Glass Works**, Corning, N.Y.

[22] Filed: **Aug. 10, 1971**

[21] Appl. No.: **170,569**

[52] **U.S. Cl.** **343/911**, 343/754

[51] **Int. Cl.** **H01q 15/08**

[58] **Field of Search** 343/754, 755, 909, 910, 911

[56] **References Cited**

UNITED STATES PATENTS

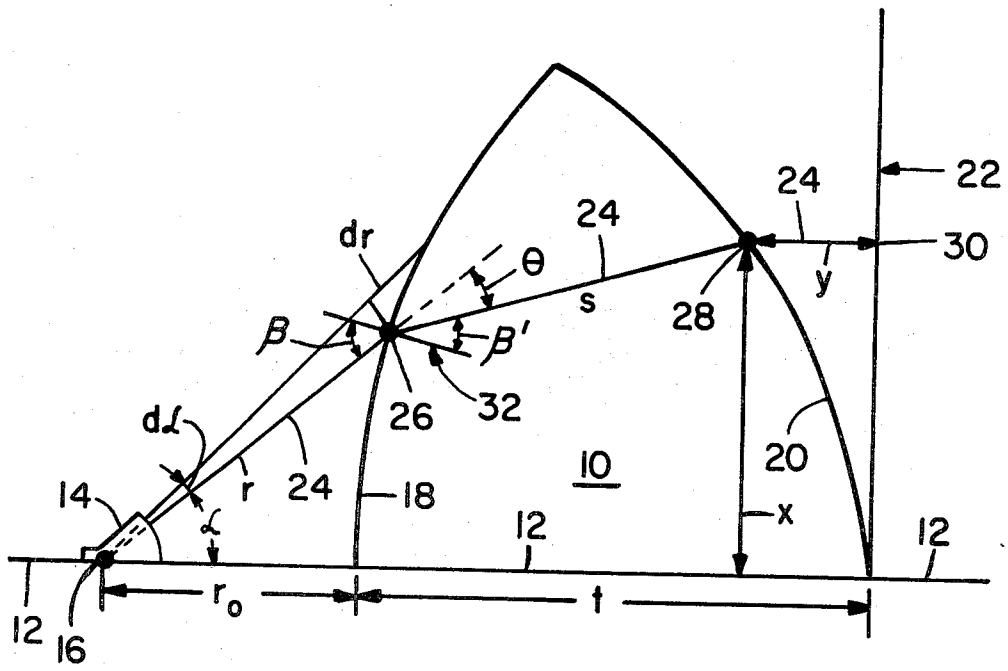
2,479,673	8/1949	Devore	333/98
2,547,416	4/1951	Skellett	343/910 X
2,761,141	8/1956	Strandberg et al.	343/911 L
2,849,713	8/1958	Robinson	343/911 L
3,256,373	6/1966	Horst	343/911 L
3,264,648	8/1966	Sundberg et al.	343/754
3,392,395	7/1968	Hannan	343/755
2,975,419	3/1961	Brown	343/754

Primary Examiner—John W. Huckert
Assistant Examiner—R. E. Hart
Attorney, Agent, or Firm—Walter S. Zebrowski et al.

[57] **ABSTRACT**

A microwave lens for use in the transmission of microwave energy, and a method of producing such a lens. The microwave lens of this invention transforms microwave energy passing through the lens to a desired specific phase and amplitude distribution. Both surfaces of the lens are contoured interdependently to provide the desired phase and amplitude distribution.

20 Claims, 15 Drawing Figures



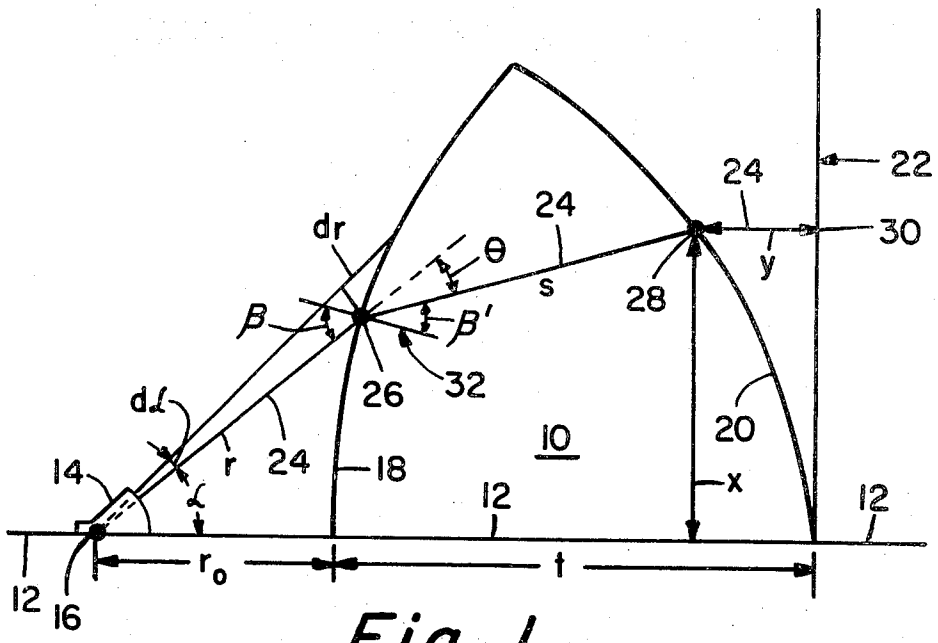

dr
θ
24
s
24
22
30
y
28
β
β'
26
32
dℒ
r
24
18
10
20
14
ℒ
12
x
12
12
16
r_0
t

Fig. 1

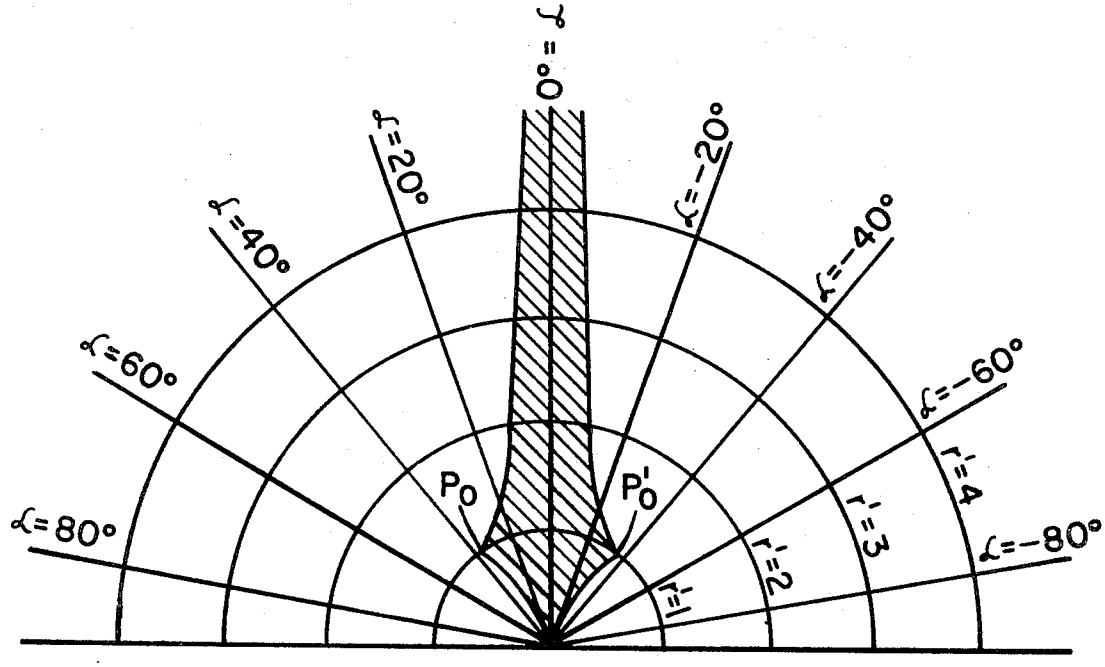

ℒ=0°
ℒ=20°
ℒ=-20°
ℒ=40°
ℒ=-40°
ℒ=60°
ℒ=-60°
ℒ=80°
ℒ=-80°
P_0
P_0'
r=1
r=2
r=3
r=4

Fig. 2

Fig. 3

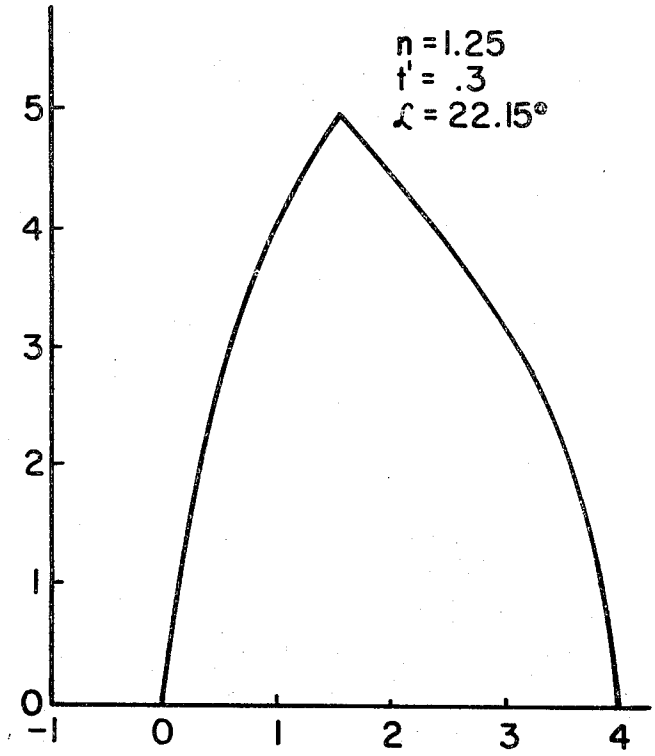


Fig. 4

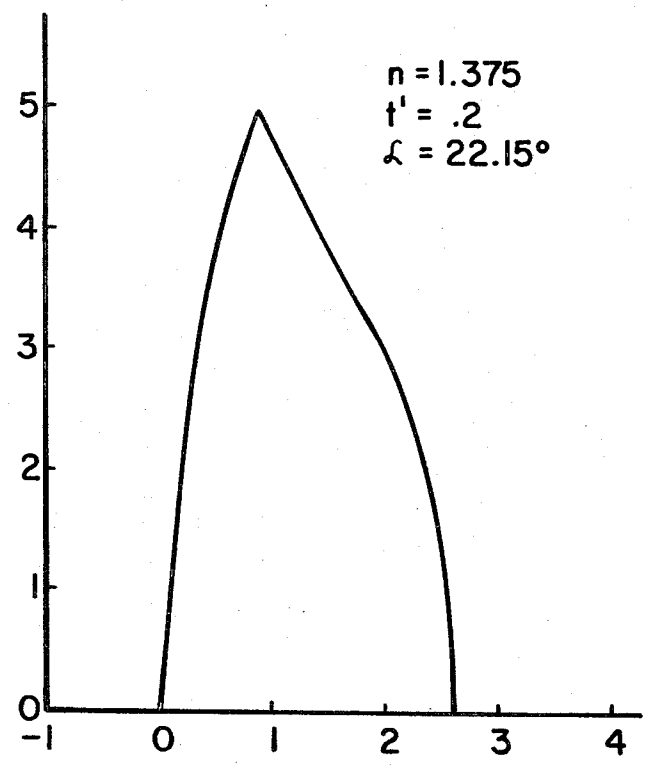


Fig. 5

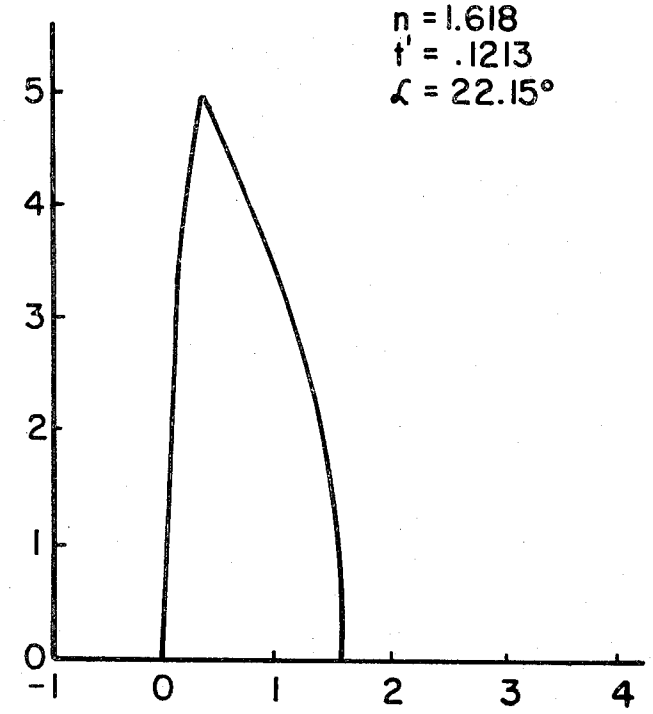

n = 1.618
t' = .1213
ℒ = 22.15°
5
4
3
2
1
0
-1
0
1
2
3
4

Fig. 6

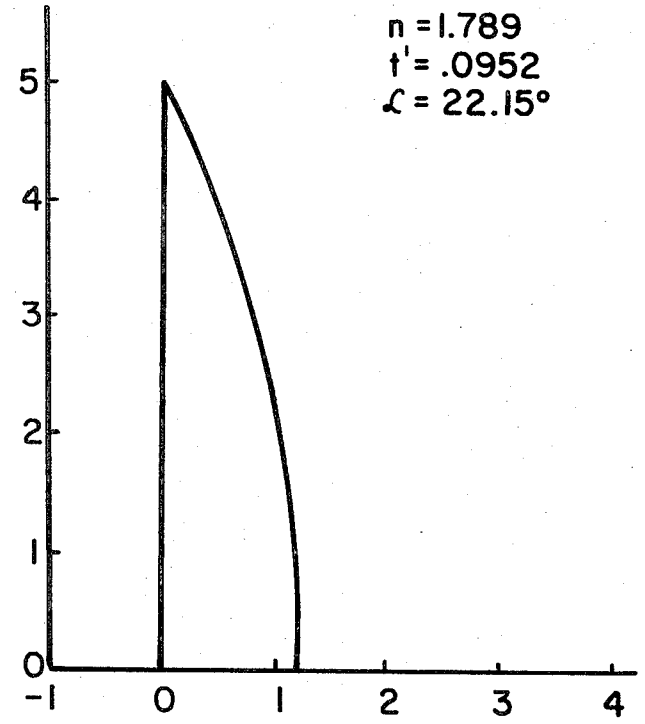

n = 1.789
t' = .0952
ℒ = 22.15°
5
4
3
2
1
0
-1
0
1
2
3
4

Fig. 7

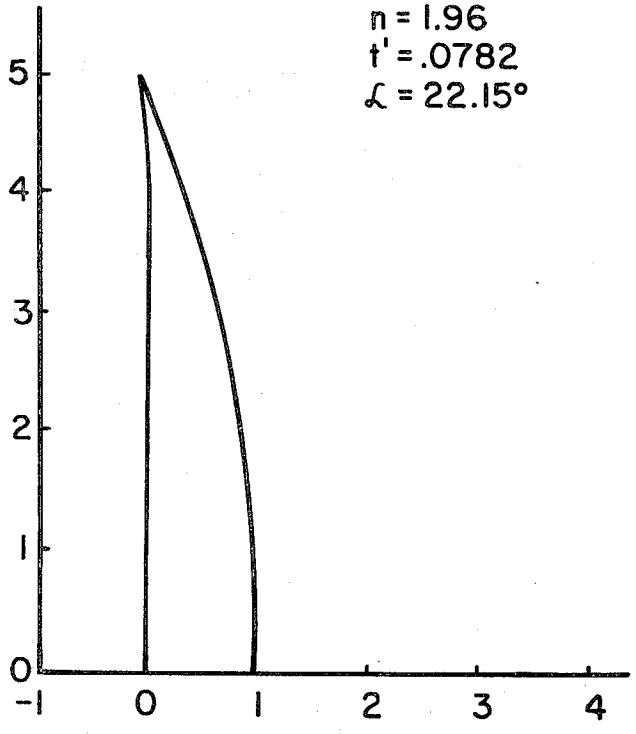
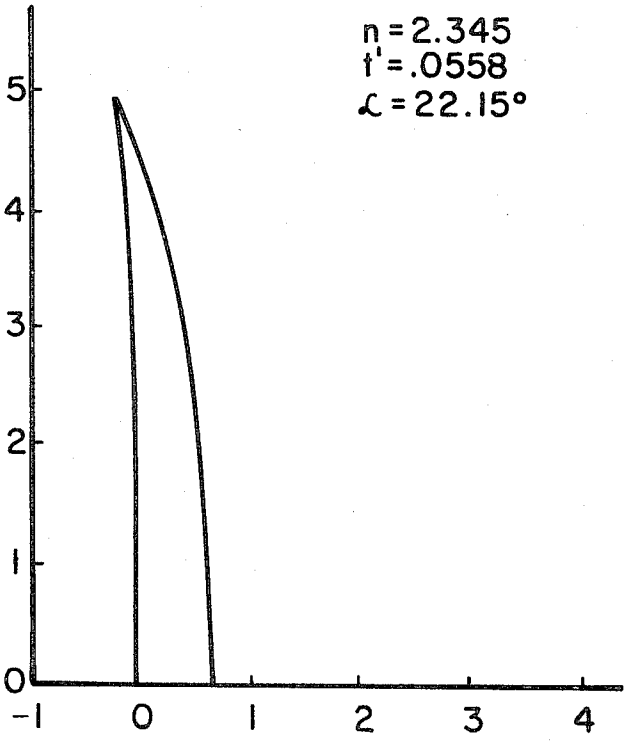

n = 1.96
t' = .0782
ℒ = 22.15°
5
4
3
2
1
0
-1
0
1
2
3
4

Fig. 8


n = 2.345
t' = .0558
ℒ = 22.15°
5
4
3
2
1
0
-1
0
1
2
3
4

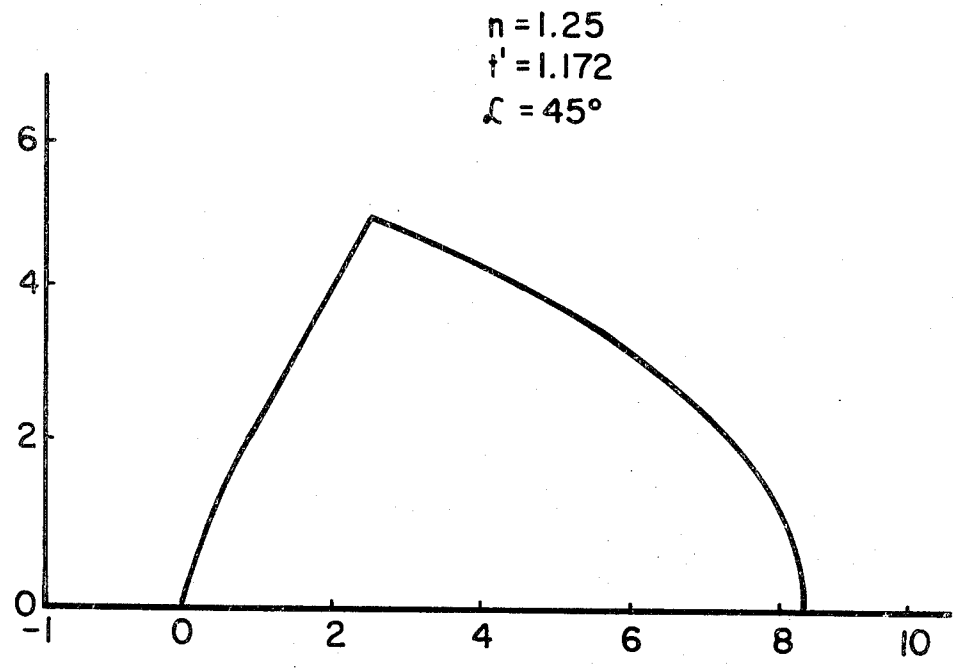

n = 1.25
t' = 1.172
ℒ = 45°
6
4
2
0
-1
0
2
4
6
8
10

Fig. 9

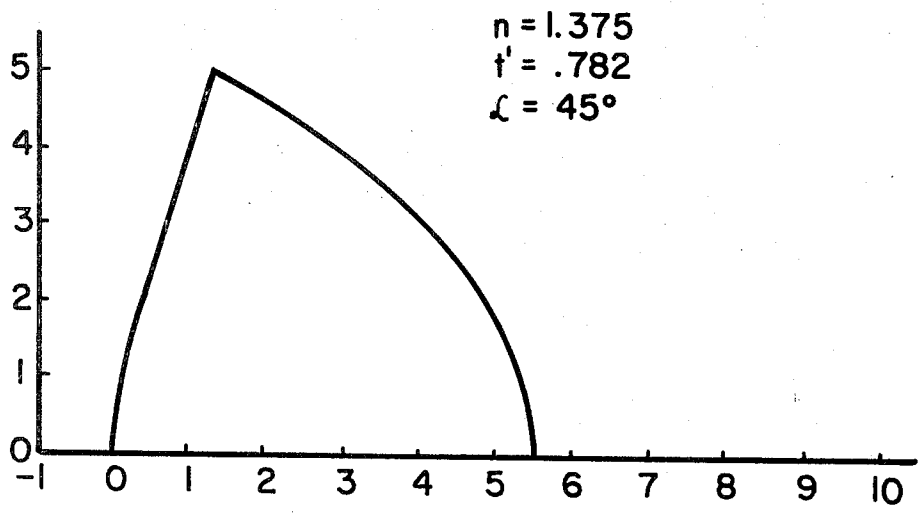

n = 1.375
t' = .782
ℒ = 45°
5
4
3
2
1
0
-1
0
1
2
3
4
5
6
7
8
9
10

Fig. 10

Fig. 11

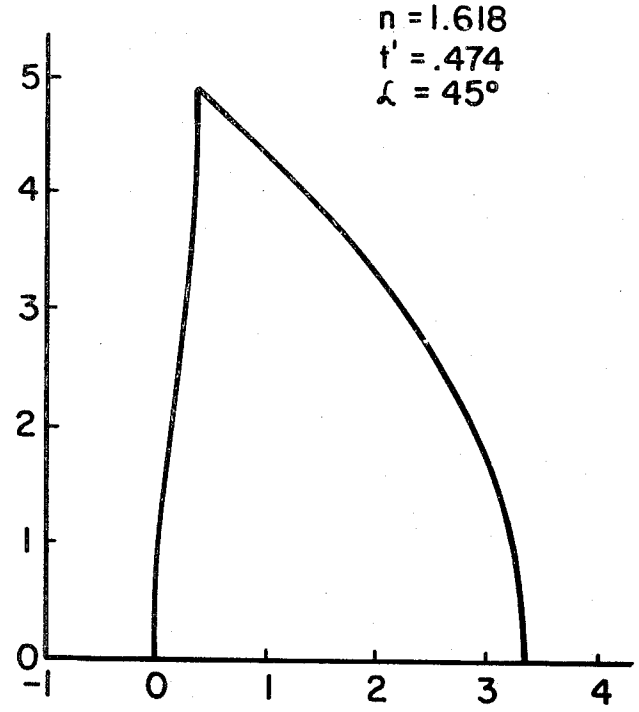


Fig. 12

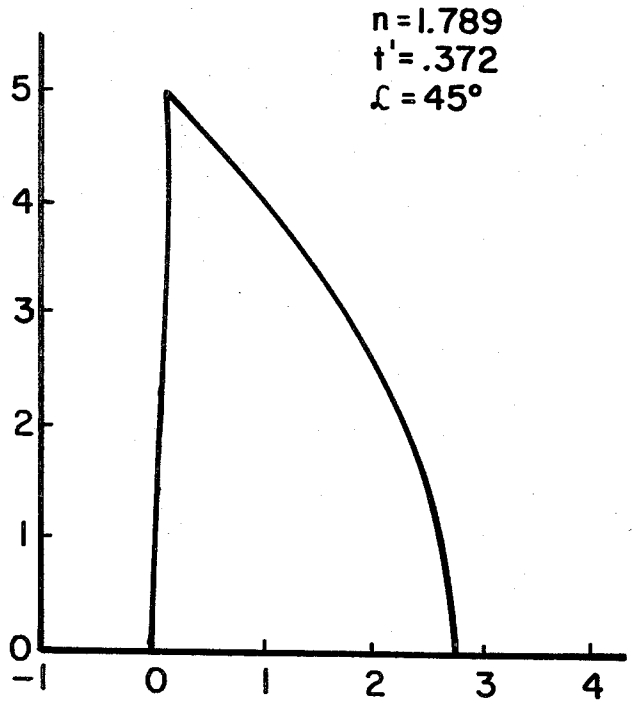


Fig. 13

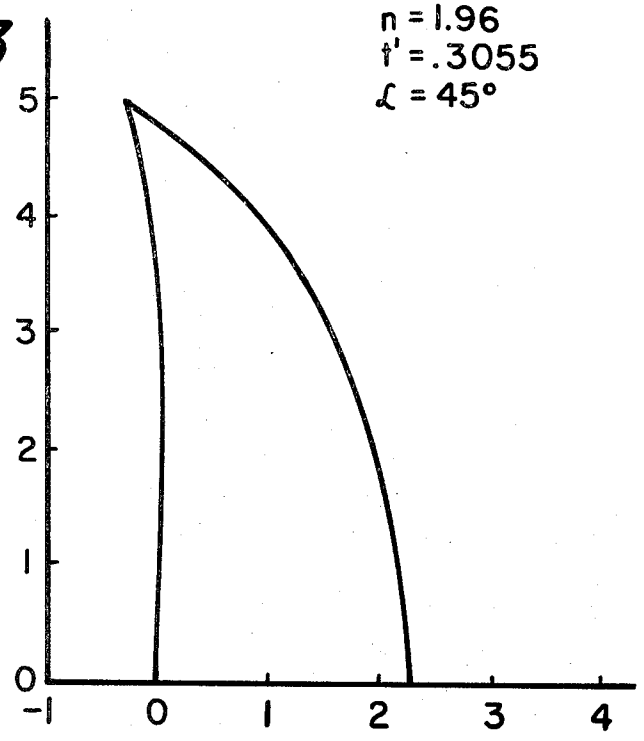
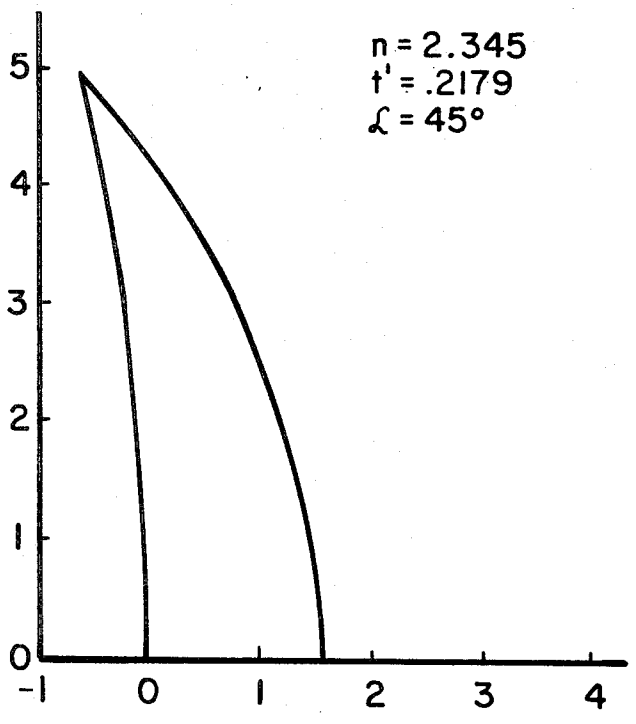


Fig. 14



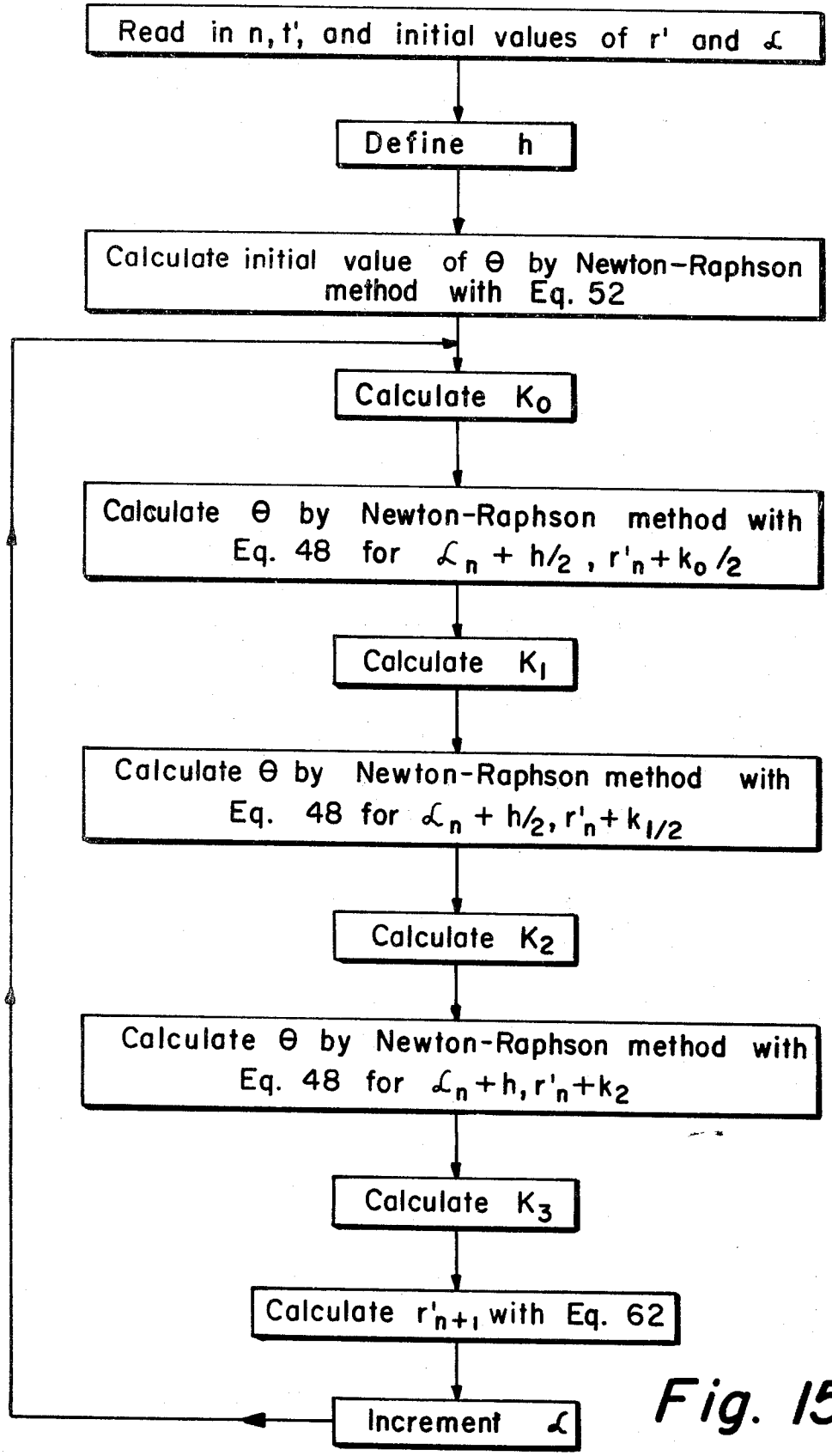

Read in n, t', and initial values of r' and ℒ
Define h
Calculate initial value of Θ by Newton-Raphson method with Eq. 52
Calculate K_0
Calculate Θ by Newton-Raphson method with Eq. 48 for $\mathcal{L}_n + h/2$, $r'_n + k_0/2$
Calculate K_1
Calculate Θ by Newton-Raphson method with Eq. 48 for $\mathcal{L}_n + h/2$, $r'_n + k_{1/2}$
Calculate K_2
Calculate Θ by Newton-Raphson method with Eq. 48 for $\mathcal{L}_n + h$, $r'_n + k_2$
Calculate K_3
Calculate r'_{n+1} with Eq. 62
Increment ℒ

Fig. 15

MICROWAVE LENS ANTENNA AND METHOD OF PRODUCING

INVENTION MADE UNDER GOVERNMENT CONTRACT

The invention herein described was made in the course of or under a contract or subcontract thereunder, with the department of the Air Force.

BACKGROUND OF THE INVENTION

This invention relates to microwave lenses used for controlling the distribution of microwave energy passing through such lenses and the method of producing such lenses. More specifically, this invention relates to microwave lenses capable of controlling both the phase and amplitude distribution of microwave energy passing through the lens by interdependently contouring the lens' two surfaces.

Lenses presently used in microwave systems are of two types: (1) Lenses having a variable index of refraction, and (2) Lenses having a constant index of refraction. The variable refractive index lenses, of which the Luneburg lens is perhaps the best known example, achieve their focusing or energy concentrating property by refraction of the energy. This refraction occurs as a result of the variation in the refractive index of the lens material. Lenses of this type, although very effective in theory, have the disadvantages of being very heavy and difficult to fabricate. The constant refractive index lenses presently available achieve their focusing and energy concentration properties by refraction of the microwave energy at one of the lens' two surfaces. These lenses are normally designed by setting an equation which defines the effective optical path length of any microwave energy ray emitted from a source located at the lens' focal point and which passes through the lens and arrives at a reference plane, equal to a constant. As is recognized by one skilled in the art, if the optical path length of all of the microwave energy rays is constant, the microwave energy will have a constant phase. Further, achieving microwave energy with a constant phase is commonly accomplished by letting one surface of the lens be flat and varying the other surface as necessary. Such constant refractive index lenses are fairly easy to fabricate. However, both the variable refractive index lenses and the constant refractive index lenses presently in use, are designed strictly according to phase distribution considerations. Amplitude distribution of microwave energy which has passed through a lens of variable index of refraction or a lens having a constant index of refraction is uncontrolled and varies widely, thereby resulting in low efficiency.

SUMMARY OF THE INVENTION

Therefore, it is an object of this invention to provide a simple economical lens, and a method of producing such a lens, for use with microwave energy systems, such lenses overcoming all of the heretofore noted disadvantages of the presently available microwave lenses.

Briefly, this invention comprises a microwave lens for use in the transmission of microwave energy and a method of production. The microwave lens is made from a dielectric material having a constant index of refraction. The exit and entry surfaces of the lens are interdependently contoured such that both the phase and amplitude distribution of microwave energy passing through the lens is controlled as desired.

Additional objects, features and advantages of the present invention will become apparent to those skilled in the art from the following detailed description and attached drawings.

BRIEF DESCRIPTION OF THE DRAWINGS

FIG. 1 is a diagrammatic illustration of a microwave lens system, and is used to aid the derivation of equations which describe the two contour surfaces of lens made in accordance with this invention.

FIG. 2 illustrates, by the shaded area of the curve, the region where solutions exist for one of the equations derived in the specification.

FIGS. 3–14 illustrate the contour of lenses having various indices of refraction and thicknesses, which contours were determined by the method of this invention.

FIG. 15 is a flow chart of a computer program used in the simultaneous solution of transcendental equations necessary for determining the contours of the surfaces of the lens of this invention.

DETAILED DESCRIPTION

As was discussed heretofore, microwave lenses capable of controlling phase distribution only of microwave energy have been used for several years; however, the amplitude distribution of the microwave energy is uncontrolled by these lenses and varies widely. This lack of control over the amplitude distribution has severely limited the use of microwave lenses. According to this invention, a novel and valuable microwave lens is disclosed which can transform microwave energy having a given power distribution at the lens focal point and passing through the lens into any desired predetermined phase and amplitude distribution. Also disclosed is a method of fabrication.

The lens of this invention has two interdependent contoured surfaces and a constant index of refraction. Although some lenses, used only for controlling phase distribution, have two contoured surfaces, most microwave lenses presently available are typically plano-convex lenses, having one contoured surface and one flat surface. However, since two surfaces may be shaped and contoured during fabrication of the lens, two physical constraints may be specified. Therefore, in the lens of this invention, in addition to the present practice of specifying the optical path lengths through the lens, which results in the desired phase distribution, the desired amplitude distribution of microwave energy which has passed through the lens is also specified. These two conditions are then used to determine the necessary interdependent contours of the two surfaces of a lens which is made of material having a particular index of refraction. The effect of the index of refraction of the lens material upon the contours of the two surfaces will be further discussed hereinafter.

The specific problem, therefore, in fabricating a lens which controls the phase and amplitude distribution of microwave energy in a predetermined and specific manner, is the determination of the necessary interdependent contours of the lens' two surfaces which will transform microwave energy received from a source located at the lens focal point and having a given power distribution $F(\alpha)$, into the desired amplitude and phase distribution. Referring now to FIG. 1, there is shown a

schematic of a lens system. Lens **10**, is rotationally symmetrical about axis **12** through the lens center. A microwave power source **14** is located at the focal point **16** of the lens. The surface nearest power source **14** is hereby designated as the first surface **18**, and the other surface is hereby designated as the second surface **20**. Line **22** represents a plane perpendicular to the center axis and tangent to the second surface **20** of the lens at the center axis. A ray of microwave energy **24** travels from focal point **16** to point **26** on surface **18** of lens **10**, is refracted by the first surface **18**, and travels through lens **10** to point **28** on surface **20**. The ray is also refracted by surface **20** as it leaves lens **10**, and travels to point **30** on line **22**. To avoid confusion in the derivation of the equations hereinafter discussed, letters rather than numbers are used to designate some of the remaining notations used on FIG. **1**. Line r represents the portion of ray **24** traveling in air from focal point **16** to point **26** on the first surface. Line r_o represents the distance along the center axis **12** from focal point **16** to the first surface. The symbol α represents the angle between the center axis **12** and line r. The symbol $d\alpha$ represents an incremental portion of angle α, and dr represents an incremental portion of line r. Line s represents the portion of ray **24** traveling through lens **10**. Line **32** is a line normal to the tangent of the lens at point **26**. The symbol β represents the angle between line **32** and line r. The symbol β' represents the angle between line **32** and line s. The symbol θ represents the angle between line r and line s, and represents the angle between the ray of microwave energy before entering lens **10** and after entering the lens. X is the perpendicular distance from center axis **12** to point **28** on surface **20** of lens **10**. Y is the perpendicular distance from line **22** to point **28** on surface **20** of lens **10**. Line t is the distance along the center axis between surface **18** and surface **20**, and represents the thickness of the lens at the center axis.

A mathematical relation involving the amplitude distribution of a lens can be obtained by writing an equation of energy conservation for the lens. For the system illustrated in FIG. **1**, this equation is

$$2\pi F(\alpha)\sin\alpha d\alpha = 2\pi I(x)\,x\,dx, \tag{1}$$

where $F(\alpha)$ in the illustration of FIG. **1** represents the amplitude distribution of microwave power at focal point **16** emitted from power source **14**. The distribution of power by $F(\alpha)$ is in the units of power per unit solid angle. The function $I(x)$ is the resulting amplitude distribution of power per unit area of plane **22**. The relation described by Equation (1) must also hold true for any increment of power passing through the lens and arriving at plane **22**. Therefore, to express an arbitrary increment of power the terms 2π are canceled, and the expression rewritten in the integrable equation

$$\int_0^{\alpha} F(\alpha)\sin\alpha d\alpha = \int_0^{x} I(x)X dx. \tag{2}$$

Therefore, after normalization with respect to the total power, the equation

$$\frac{\int_0^{\alpha} F(\alpha)\sin\alpha d\alpha}{\int_0^{\alpha_{max}} F(\alpha)\sin\alpha d\alpha} = \frac{\int_0^{x} I(x)X dx}{\int_0^{x_{max}} I(x)X dx} \tag{3}$$

representing amplitude distribution across a lens is obtained, where $F(\alpha)$ and $I(x)$ must be integrable functions.

An equation for expressing phase distribution of microwave energy passing through a lens may be obtained by setting the optical path length of rays of microwave energy passing through the lens equal to the desired phase distribution $\phi(x)$ at plane **22**. The general expression for a segment of optical path length is equal to $\int nds$, where n is the index of refraction of the medium and ds is an increment of arc length along the ray.

Therefore, by setting the expression equal to the desired phase function, the equation

$$\phi(x) = \int nds \text{ is obtained.} \tag{4}$$

For the lens system illustrated in FIG. **1**, the optical path length between focal point **16** and point **30** on reference plane **22** may be expressed as

$$n_{air}r + n_{lens}s + n_{air}Y = \phi(x). \tag{5}$$

However, since the index of refraction of air is 1, Equation (5) becomes

$$r + n_{lens}s + Y = \phi(x). \tag{6}$$

The contours of the lens' two surfaces are interdependent and are determined by phase and amplitude consideration. In the system illustrated by FIG. **1**, the first surface **18** may be expressed in terms of the variables r and α, and the second surface **20** may be expressed in terms of the variables X and Y. Therefore, it is necessary that equations (3) and (6) be related, and that other equations containing r, α, X and Y be derived. The appropriate interdependency and the necessary equations can be obtained by proceeding as follows.

Additional geometrical relations applicable to the lens system illustrated in FIG. **1** and which are needed to determine the two contoured surfaces of the lens are:

the differential equation of the first surface which is

$$1/r\ dr/d\alpha = \tan\beta; \tag{7}$$

Snell's Law which for the system shown in FIG. **1** can be expressed as

$$\sin\beta = n\sin\beta'; \tag{8}$$

and the equations

$$X = r\sin\alpha + s\sin(\alpha-\theta); \tag{9}$$

$$\beta = \beta' + \theta; \tag{10}$$

and

$$r\cos\alpha + s\cos(\alpha-\theta) + Y = r_o + t. \tag{11}$$

The X and Y coordinates of the second surface **20** can be obtained in terms of the variable parameters r and α, angle θ, and the phase relation of equation (6), by first solving Equation (9) for s to obtain

$$s = [X - r\sin\alpha/\sin(\alpha-\theta).] \tag{12}$$

Equation (12) is then substituted into equation (6) and the resulting equation rewritten as

$$[n/\sin(\alpha-\theta)]\,X + Y = \phi(x) - r + [rn\sin\alpha/\sin(\alpha-\phi)] \tag{13}$$

Equation (12) is also substituted into Equation (11) and the resulting equation rewritten to yield

$$[\cos(\alpha-\theta)/\sin(\alpha-\theta)]\ X + Y = r_o + t - r\cos\alpha + [r\sin\alpha\cos(\alpha-\theta)/\sin(\alpha-\theta)]. \quad (14)$$

From equations (13) and (14), expressions for X and Y, the coordinates of the second surface which are related to the phase distribution of power through a lens, can be written as

(15)

$$X - r\sin\alpha + \left(\frac{\varphi(x) - r_o - t - r(1-\cos\alpha)}{n-\cos(\alpha-\theta)}\right)\sin(\alpha-\theta)$$

and

(16)

$$Y = \frac{n(r_o + t + \cos\alpha) - (\varphi(x) - r)\cos(\alpha-\theta)}{n-\cos(\alpha-\theta)}.$$

Integrating the right hand side of equation (3) we obtain

$$(17)\quad \frac{\int_0^{\alpha} F(\alpha)\sin\alpha d\alpha}{\int_0^{\varphi_{max}} F(\alpha)\sin\alpha d\alpha} = \frac{F(x)}{K_{(x_{max})}},$$

where $F(x)$ represents the value of

$$\int_0^x I(x)Xdx$$

after integrating and $K_{(x_{max})}$ represents the value of

$$\int_0^{x_{max}} I(x)Xdx$$

after integrating and evaluating. $K_{x_{max}}$ will be a constant.

Therefore, Equation (17) can be rewritten as the general equation

$$(18)\quad F(x) = K_{(x_{max})}\frac{\int_0^{\alpha} F(\alpha)\sin\alpha d\alpha}{\int_0^{\alpha_{max}} F(\alpha)\sin\alpha d\alpha}.$$

Equation (8) and equation (10) can be combined to give equation

$$\sin\beta = n\sin(\beta-\theta). \quad (19)$$

Equation (19) can then be expanded to

$$\sin\beta = (\sin\beta\cos\theta - \cos\beta\sin\theta). \quad (20)$$

Equation (20) can be rewritten as

$$\sin\beta(n\cos\theta-1) = n\sin\theta\cos\beta. \quad (21)$$

and finally equation (21) can be rewritten as

$$\tan\beta = n\sin\theta/n\cos\theta-1. \quad (22)$$

Combining equation (22) with equation (7) yields

$$dr/d\alpha = rn\sin\theta/n\cos\theta-1 \quad (23)$$

Equations (15), (16), (18) and (23) are the general relations for determining the contours of a lens built in accordance with this invention. The specific desired phase distribution $\phi(x)$ and amplitude distribution $I(x)$ must be substituted in these equations before they can be used to calculate lens contours to produce the desired distributions. Also, the feed power distribution $F(\alpha)$ must be chosen to make the equations complete.

Maximum gain of an antenna is achieved when both amplitude distribution and phase distribution are uniform or constant. Therefore, in the following preferred embodiment, equations (15), (16), (18) and (23) are further developed for the specific case of determining the contour of the two surfaces for a microwave lens which will provide a substantially uniform phase and uniform amplitude distribution. That is, $\phi(x)$ and $I(x)$ are constant. For either phase or amplitude distributions other than uniform distribution, an integrable function expressing the desired variations must be substituted for $\phi(x)$ and $I(x)$ in the equations, and the equations developed in a manner similar to that described in the following preferred embodiment.

PREFERRED EMBODIMENT

For a lens having uniform phase distribution, $\phi(x)$ as expressed by equation (6) is set equal to a constant, such that equation (6) becomes

$$r + n_{lens}s + Y + K_1, \quad (24)$$

and from FIG. 1 it can be seen that the constant K_1 may be chosen to equal the optical path length of the center ray such that equation (24) becomes

$$r + ns + Y = r_o + nt, \quad (25)$$

where n equals n_{lens}, r_o equals the distance along the center axis **12** between focal point **16** and the lens first surface **18**, and t equals the distance along the center axis **12** between the lens first surface **18** and the second surface **20**. Therefore,

$$\phi(x) = r_o + nt. \quad (26)$$

Substituting $r_o + nt$ for $\phi(x)$ into equations (15) and (16), rearranging and cancelling terms we obtain new equations for X and Y

$$(27)\quad X = r\sin\alpha + \frac{t(n-1) - r(1-\cos\alpha)}{n-\cos(\alpha-\theta)}\sin(\alpha-\theta)$$

$$(28)\quad y = \frac{n(r_o + t + \cos\alpha) + (r - r_o + nt)\cos(\alpha-\theta)}{n-\cos(\alpha-\theta)}$$

In a like manner, if uniform amplitude distribution is to be achieved $I(x)$ of equation (3) must also be a constant. Therefore, $[F(x)/K_{(x}$ $_{)}]$ of equation (17) becomes X^2/X^2_{max} and we obtain

$$(29)\quad \frac{\int_0^{\alpha} F(\alpha)\sin\alpha d\alpha}{\int_0^{\alpha_{max}} F(\alpha)\sin\alpha d\alpha} = \frac{X^2}{X^2_{max}}.$$

Taking the square root of equation (29) and rearranging yields the equation

$$(30)\quad X = X_{max}\left(\frac{\int_0^{\alpha} F(\alpha)\sin\alpha d\alpha}{\int_0^{\alpha_{max}} F(\alpha)\sin\alpha d\alpha}\right)^{1/2}.$$

Normally the distribution of power from power source **12** is preferably isotropic over the solid angle subtended by the lens. That is $F(\alpha)$ is a constant. Therefore, if $F(\alpha)$ is a constant it can be put outside the integral and eliminated. Therefore, equation (30) becomes

$$X = X_{max}\left(\frac{\int_0^{\alpha} \sin \alpha d\alpha}{\int_0^{\alpha_{max}} \sin \alpha d\alpha}\right)^{1/2}. \tag{31}$$

Substituting the value of X as expressed by equation (27), we obtain

$$r \sin \alpha + \frac{t(n-1)-r(1-\cos \alpha)}{n-\cos(\alpha-\theta)} \sin(\alpha-\theta) = X_{max}\left(\frac{\int_0^{\alpha} \sin \alpha d\alpha}{\int_0^{\alpha_{max}} \sin \alpha d\alpha}\right)^{1/2}. \tag{32}$$

Therefore, we have the four equations (23), (27), (28) and (32) which when solved will describe a microwave lens having constant phase and constant amplitude, in terms of the two variables r and α for the first surface, and the two variables X and Y for the second surface.

Therefore, obtaining a solution for equations (23) and (32) will determine the first surface.

In order to integrate equation (23), an initial point is needed. The initial point may be obtained by proceeding as follows. By using equation (32), the region in which the integral curves of equation (23) lie for given values of n and t can be derived. Rearranging equation (32), and performing the integration gives equation

$$\frac{\sin(\alpha-\theta)}{n-\cos(\alpha-\theta)} = \frac{X_{max}}{\frac{\sqrt{1-\cos \alpha_{max}}\sqrt{1-\cos \alpha}-r \sin \alpha}{t(n-1)-r(1-\cos \alpha)}}, \tag{33}$$

The extreme values of the left hand side of equation (33) may be found by differentiating with respect to α and setting the derivative equal to zero. Performing these operations yields:

$$\frac{d}{d\alpha}\left(\frac{\sin(\alpha-\theta)}{n=\cos(\alpha-\theta)}\right) = \frac{n\cos(\alpha-\theta)-1}{(n-\cos(\alpha-\theta))^2} = 0 \tag{34}$$

Rearranging and cancelling terms gives the equation

$$\cos(\alpha-\theta) = 1/n. \tag{35}$$

From equation (35) the equation

$$\sin(\alpha-\theta) = \sqrt{n^2-1}/n \text{ can be determined.} \tag{36}$$

Therefore,

$$\left|\frac{\sin(\alpha-\theta)}{n-\cos(\alpha-\theta)}\right| \leq \pm \frac{1}{\sqrt{n^2-1}}. \tag{37}$$

Substituting the value of equation (37) back into equation (33) yields the equation

$$\frac{1}{\sqrt{n^2-1}} \leq \frac{X_{max}\sqrt{1-\cos \alpha}-r \sin \alpha}{\sqrt{1-\cos \alpha_{max}}\, t(n-1)-r(1-\cos \alpha)} \leq \frac{1}{\sqrt{n^2-1}} \tag{38}$$

Equations (16) and (25) are also used to help determine the initial point. Eliminating Y between equations (16) and (25) yields the equation

$$r(1-\cos\alpha) + s(n-\cos(\alpha-\theta) = t(n-1). \tag{39}$$

Referring to FIG. **1**, and using equation (39), it can be seen that for $s \geq 0$, equation (39) becomes $r(1-\cos\alpha) \leq (n-1)t$. (40)

Therefore, from equation (38) and (40) it can be shown that

$$0 \leq r \leq \frac{\frac{X_{max}}{\sqrt{1-\cos \alpha_{max}}}\sqrt{(n^2-1)(1-\cos \alpha)}+t(n-1)}{\sqrt{n^2-1}\sin \alpha+1-\cos \alpha} \tag{41}$$

for

$$\alpha \leq \cos^{-1}\left(1-\frac{t^2(n-1)(1-\cos \alpha_{max})}{X_{max}^2(n+1)}\right) \tag{42}$$

and

$$\frac{\frac{X_{max}\sqrt{(n^2-1)(1-\cos \alpha)}-t(n-1)}{\sqrt{1-\cos \alpha_{max}}}}{\sqrt{n^2-1}\sin \alpha+\cos \alpha-1} \leq r \leq \frac{\frac{X_{max}}{\sqrt{1-\cos \alpha_{max}}}\sqrt{(n^2-1)(1-\cos \alpha)}+t(n-1)}{\sqrt{n^2-1}\sin \alpha-\cos \alpha+1} \tag{43}$$

for

$$\alpha \geq \cos^{-1}\left(1-\frac{t^2(n-1)(1-\cos \alpha_{max})}{X_{max}^2(n+1)}\right) \tag{44}$$

The integral curves of equation (23) are contained within a region determined by equations (41), (42), (43) and (44).

Referring again to FIG. **1** and equation (39), it can be seen that as the distance s between the first and second surface approaches zero, equation (39) becomes

$$r(1-\cos\alpha) = t(n-1) \tag{45}$$

If δ is the value of α at which the two lens surfaces meet and r_δ is the value of r when $\alpha = \delta$, equation (45) becomes

$$r_\delta\ (1-\cos\delta) = t(n-1). \tag{46}$$

By normalizing equation (46) with respect to r_δ and rewriting, an equation for δ can be written as

$$\delta = \cos^{-1}(1-t'(n-1)) \tag{47}$$

where $t' = t/r_\delta$.

An interesting aspect of equation (47) is that the angle δ is the maximum value of α, and consequently determines the maximum obtainable lens radius for given values of t' and n, since this is the angle at which the two surfaces meet. Thus, the required initial point for equation (23) in normalized coordinates is 1, δ.

The value of θ is also needed at this initial point. This initial value of θ may be obtained by proceeding from equation (33). Normalizing equation (33) with respect to r_δ and substituting δ for α_{max} gives the equation

$$\frac{\sin(\alpha-\theta)}{n-\cos(\alpha-\theta)} = \frac{\frac{X'_{max}}{\sqrt{1-\cos\delta}}\sqrt{1-\cos\alpha} - r'\sin\alpha}{t'(n-1)-r'(1-\cos\alpha)} \tag{48}$$

where $r' = r/r_\delta$, $t' = t/r_\delta$, and $X'_{max} = X_{max}/r_\delta$. *It should also be noted, that* X_{max} equals $r \sin\delta$.

Taking the limit of equation (48) as $\alpha \to \phi$ δ and $r' \to$ 271 yields

$$\sin(\delta-\theta)/n-\cos(\delta-\theta) \to 0/0, \text{ an indeterminate form.} \tag{49}$$

However, letting θ_o be the value of θ at $\alpha = \delta$, and applying L' Hospital's rule to the right hand side of equation (48) results in the equation

$$\frac{\sin(\delta-\theta_0)}{n-\cos(\delta-\theta_o)_{\alpha-\delta}} = \text{limit} \frac{2\frac{X'_{max}}{\sqrt{1-\cos\delta}}\sqrt{\frac{\sin}{1-\cos\alpha}} - r'\cos\alpha - \sin\alpha\frac{dr'}{d\alpha}}{-r'\sin\alpha + \frac{dr'}{d\alpha}(\cos\alpha-1)} \tag{50}$$

Normalizing equation (23) with respect to r_δ results in the equation

$$dr'/d\alpha = r'\, n \sin\ /n\cos\theta - 1. \tag{51}$$

Substituting equation (51) into equation (50) and taking the limit as $\alpha \to \alpha$ and $r' \to 1$ the equation

$$\frac{\sin(\alpha-\theta_o)}{n-\cos(\delta-\theta_o)} = \frac{\frac{X'_{max}}{2}\frac{\sin\delta}{(1-\cos\delta)} - \cos\delta - \frac{\sin\delta n \sin\theta}{n\cos\theta_o - 1^o}}{\frac{n\sin\theta_o}{n\cos\theta_o} - 1\,(\cos\delta-1) - \sin\delta} \tag{52}$$

is obtained from which θ_0 the initial value of θ may be found.

As was mentioned heretofore, the integral curves of Equation (23) are contained within a region determined by Equations (41), (42), (43) and (44). Normalizing these equations with respect to r_δ yields:

$$0 \le r' \le \frac{\sqrt{(n^2-1)(2-[n-1]t')(1-\cos\alpha)} + (n-1)t'}{\sqrt{n^2-1}\sin\alpha - \cos\alpha + 1} \tag{53}$$

for

$$\alpha \le \cos^{-1}\left(1 - \frac{(n-1)t'^2}{[2-(n-1)t'][n+1]}\right) \tag{54}$$

and

$$\frac{\sqrt{(2-[n-1]t')(n^2-1)(1-\cos\alpha)} - (n-1)t'}{\sqrt{n^2-1}\sin\alpha + \cos\alpha - 1} \le r' \le \frac{\sqrt{(2-[n-1]t')(n^2-1)(1-\cos\alpha)}}{\sqrt{n^2-1}\sin\alpha - \cos\alpha + 1} + (n-1)t' \tag{55}$$

for

$$\alpha \ge \cos^{-1}\left(1 - \frac{(n-1)t'^2}{[2-(n-1)t'][n+1]}\right). \tag{56}$$

For purposes of illustration, a sketch of the region determined by equations (53), (54), (55) and (56) (where $n = 1.414$ and $t' = 0.5$) is shown by the shaded area in FIG. 2. The lens of maximum radius, for the values of n and t' given above may be determined from the values of r and α at the points P_o and P'_o shown in FIG. 2.

A continuous solution to equation (24) will not exist unless

$$f[\theta(r,\alpha)] = r\, n \sin\theta/n\cos\theta - 1 \tag{57}$$

is a continuous function in the region of interest. Notice that $f[\theta(r,\alpha)]$ *has a singularity at the value of* θ *given by*

$$\theta_s = \cos^{-1} 1/n. \tag{58}$$

Thus, if n and δ are chosen such that $\theta_o > \theta_s$, the function $f[\theta(r,\alpha)]$ *cannot be determined in the* θ*-interval in which the solution must lie. For this case, the region given by equations* (52) through (55) are undefined, and no solution of equation (23) can be obtained. This situation arises when large values of aperture angle α and small values of refractive index n are chosen.

Therefore, to determine the contours of the two surfaces of a microwave lens fed from a power source having isotropic power distribution over the solid angle subtended by the lens, and having an output of constant power and phase distribution, simultaneous solutions to equations (23) and (32) are needed. The solutions of these two equations being subject to an initial condition which may be obtained by equation (52). The solutions of equations (23) and (32) will determine the contour of the first surface. The contour of the second surface is then found by using equations (27) and (28). Equations (23), (32) and (52) are non linear, and therefore require numerical solutions. In the specific example hereinafter discussed, a numerical method of solving equations (23), (32) and (52) is set out.

FIGS. 3 through 14 illustrate the surface contours of twelve 5.00 inch radius constant phase and constant amplitude lenses determined in accordance with this invention. FIGS. 3 through 8 illustrate the calculated surface contours of lenses having an arbitrarily se-

lected angle δ of 44.3°. FIGS. 9 through 14 illustrate the calculated surface contours of lenses having an arbitrarily selected angle δ or 90°. Table I sets out the parameters n, t' and δ used to calculate the contours of the 5.00 inch radius lenses illustrated by FIGS. 3 through 14. Thus lenses which perform the same focusing may be evaluated with regard to other characteristics.

TABLE I

Illustrated Figure	Subtended Angle	Index of Refraction (n) at Microwave Frequencies	Normalized thickness (t')	Lens Radius (inches)
3	22.15°	1.25	0.3	5.00
4	22.15°	1.375	0.2	5.00
5	22.15°	1.618	0.1213	5.00
6	22.15°	1.789	0.0952	5.00
7	22.15°	1.96	0.0782	5.00
8	22.15°	2.345	0.0558	5.00
9	45°	1.25	1.172	5.00
10	45°	1.375	0.782	5.00
11	45°	1.618	0.474	5.00
12	45°	1.789	0.372	5.00
13	45°	1.96	0.3055	5.00
14	45°	2.345	0.2179	5.00

An evaluation of all the contours illustrated by FIGS. 3–14 show that the lens thickness varies inversely with the refractive index of the material if the focal point remains the same. Further, as the refractive index is increased the surface goes from convex to concave, and for an intermediate value of $n = 1.789$, the contour of one of the surfaces is substantially planar as is shown in FIGS. 6 and 12. A lens having one planar or flat surface is of considerable interest, as the fabrication cost of a lens with a flat surface is much lower than the cost of fabricating a two contoured surface lens. It should be noted, that the index of refraction of a material varies as the frequency varies. Therefore, the particular value of the index of refraction given for a particular material is the index of refraction of that material at microwave frequencies.

Although lenses built in accordance with the teachings of this invention could be fabricated from any material with the desired index of refraction, including various plastics, it has been found that glass, glass-ceramics and ceramics are particularly suitable materials. The suitability of such materials for this use is primarily a result of the superior thermal properties of the materials.

In the construction of conventional antennas, suitable dimension tolerances may be achieved under ideal conditions. However, maintaining these tolerances under the temperature extremes of normal operating conditions is somewhat difficult. The low coefficient of thermal expansion of glass, glass-ceramics or ceramics greatly diminishes the problem of thermal distortion. Although in some cases the thickness of the lens may result in more weight, such a lens has the offsetting advantage that this thickness offers rigidity without the necessity of a supporting backup structure normally used with conventional antennas. This elimination of backup structure further reduces the thermal distortion. In addition, the materials glass, glass-ceramic and ceramic have a very low loss tangent, and the index of refraction variation with frequency is very small over the microwave frequency range. This last point is of considerable importance, since it means that a lens made from these materials will not limit the band width of the antenna. For example, in a case of a horn-lens combination, the band width of the antenna is limited only by the band width of the feed horn.

Another desirable property is that these materials will not deteriorate with the exposure to the sun as do many other materials. Also, these materials with the exception of some foam glasses are impervious to moisture. Therefore, the use of lenses made of such material may eliminate the need for a radome. Some specific materials which are considered well suited for use with this invention are listed in Table II below. As can be seen from Table II multiform fused silica is an excellent material. It has low thermal expansion and can be formed in a precision mold thereby eliminating costly machining operations. It can also be made with an index of refraction of approximately 1.789 which as was discussed heretofore allows the fabrication of a constant phase and amplitude lens with one flat surface. In addition, the material is opaque to sunlight, which eliminates the possibility of damaging the antenna by pointing it to the sun. Table II below sets out some of the characteristics of lenses made from various suitable materials. All of the lens have a diameter of 18 inches and identical focusing characteristics.

TABLE II

Material	Weight (lbs.)	Thickness at center (inches)	Index of refraction (n) at microwave frequencies	Loss tangent	Density (lbs./ft.^{3})	Opaque to visible light
Fused silica	18.9	1.86	1.96	2×10^{-5}	137	No.
Corning Glass Works multiform fused silica	20.75	2.27	1.789	7×10^{-4}	124.7	Yes.
Corning Glass Works foam glass Code 7928	20.2	2.89	1.62	2.8×10^{-4}	93.5	Yes.
Corning Glass Works glass ceramic Code 9606	16	1.33	2.81	3.3×10^{-4}	162.3	Yes.

SPECIFIC EXAMPLE

The required surface contours of a constant phase and constant amplitude lens having a 5.00 inch radius, an index of refraction of 1.25, an angle δ of 22.15° and a value of t' equal to 0.3 is determined by this example. Solutions for θ_o and θ are obtained at selected values of between 0° and 22.15° by the Newton-Raphson iterative method. This method is described in the book by F. B. Hildebrand titled "Introduction to Numerical Analysis" and published by McGraw-Hill in 1956. To use this method, equation (57) is brought into the form

$$f(\theta) = 0. \tag{59}$$

The equation

$$\theta_{k+1} = \theta - f(\theta_k)/f'(\theta_k), \tag{60}$$

where the prime denotes differentiation with respect to θ, is used for the iteration which is continued until it converges for a solution of equation (59). It can be seen from the geometry, that θ must lie between 0 and $\pi/2$. When a trial value of θ was chosen in this range, the method converged quite rapidly, usually within four or five iterations.

The first-order differential equation was solved with the use of a fourth-order Runge-Kutta integration process. This process is described in Volume 2 of the work by I.S. Berezin and N.P. Zhidkov titled "Computing Methods," and published by Pergamon Press in London in 1965. The error per unit step-length is of order h^5, where h is the increment in the independent variable. Formulas which yield higher orders of accuracy are quite cumbersome and difficult to work with and are not necessary, since the accuracy provided by the fourth-order formula is quite good. A particular advantage of the Runge-Kutta process is that it requires knowledge only of the initial point, while most other methods require the evaluation of several higher-order derivatives at the initial point. Equation (23) can be written symbolically as

$$dr'/d\alpha = f(r', \alpha) \tag{61}$$

where the dependence of $f(r', \alpha)$ on r' and α is on the angle θ defined implicitly by equation (48). The fourth-order Runge-Kutta formula which was used for Equation (61) is

$$r'_{n+1} = r'_n + 1/6[k_o + 2k_1 + 2k_2 + k_3], \tag{62}$$

where

$$k_o = hf(r'_n, \alpha_n),$$
$$k_1 = hf(r'_n + K_{o/2}, \alpha_n + h/2)$$
$$k_2 = hf(r'_n + k_{1/2} \quad \alpha_n + h/2)$$
$$k_3 = hf(r'_n + k_2, \alpha_n + h)$$

and

$$h = \alpha_{n+1} - \alpha_n.$$

The calculation of each k requires the evaluation of $f(r', \alpha)$ for a different value of r' and α. This, in turn, requires the solution of equation (48) for these values of r and α. The program for calculating the first lens surface is outlined in the brief flow chart of FIG. **15.** The repeated calculation of θ for each of the k's was accomplished by writing a subroutine for solving equation (48) by the Newton-Raphson method which was called at the appropriate times during the execution of the program. The coordinates of the first surface are then used as input data necessary to calculate the contour of the second surface using equations (27) and (28).

Random values of θ and θ_o were checked for accuracy by substitution back into equations (48) and (52) written in the form of equation (59). In each case, the value of $f(\theta)$ or $f(\theta_o)$ was less than 10^{-6}, indicating that the calculated value was indeed a root of the equation. The stability of the solution to equation (23) obtained with the Runge-Kutta formula was checked by using some point on the contour other than the initial point as the starting point and running the iteration backwards. This is done simply by reversing the sign on the increment of the independent variable. In each check case, exactly the same contour points were obtained with the iteration proceeding in either direction. The calculations were carried out on a General Electric 265 time-shared computer. FIG. 3 illustrates the contour of the lens' two surfaces as determined by the above described iterative method. The contours of the other lens set out in Table I and illustrated by FIGS. **4–14** were also determined by the above described iterative method.

Although the present invention has been described with respect to specific details and certain embodiments thereof, it is not intended that such details be limitations upon the scope of the invention except insofar as is set forth in the following claims.

I claim:

1. A lens for providing microwave energy having predetermined desired phase and amplitude distribution, said microwave lens being made of a dielectric material having a constant index of refraction, said lens having first and second interdependent contoured surfaces for refracting microwave energy, both of said surfaces being rotationally symmetrical about a center axis, said lens being adapted to receive energy from a microwave source, said source being located at the focal point of said lens, said first surface of said lens, which first surface is the nearest surface to said source, being defined by the parameters r, which is the distance from said focal point to said first surface and which represents the line of travel of a ray of microwave energy from said focal point to said first surface and, α, which represents the angle between r and said center axis, and the second surface thereof being defined by the parameters, x which represents the perpendicular distance from said center axis to said second surface, and y, which represents the perpendicular distance from a reference plane to said second surface, said reference plane being tangent to said second surface at said center axis, said first and second surfaces further being characterized in that said parameters r, α, x and y are determined by solving the equations

$$X = r \sin \alpha + \frac{(\varphi(x) - r_o - t - r(1 - \cos \alpha) \sin (\alpha - \theta)}{n - \cos (\alpha - \theta)},$$

$$Y = \frac{n(r_o + t + \cos \alpha) - (\varphi(x) - r) \cos (\alpha - \theta)}{n - \cos (\alpha - \theta)},$$

$$\frac{dr}{d\alpha} = \frac{rn \sin \theta}{n \cos \theta - 1},$$

and

$$F(x) = \bar{K}_{(x_{max})} \frac{\int_0^{\alpha} F(\alpha) \sin \alpha d\alpha}{\int_0^{\alpha_{max}} F(\alpha) \sin \alpha d\alpha}$$

wherein

$\phi(x)$ is a function of x representing the desired phase distribution of power at said reference plane,

$F(\alpha)$ is a function of α representing the initial power distribution from said microwave source per unit angle,

r_o is the distance along the center axis from said focal point to said first surface,

t is the distance along the center axis between said first surface and said second surface,

n is the index of refraction of the material from which the lens is made,

θ is the angle between the line of travel of a ray of microwave energy before said ray enters said lens at said first surface, and after said ray enters said lens,

$F_{(x)}$ is a function of X derived by integrating

$$\int_0^{\alpha} I(x)x\,dx,$$

and

$K_{(x_{max})}$ is a constant derived by integrating and evaluating

$$\int_o^{\alpha_{max}} I(x)x\,dx,$$

wherein

$I(x)$ is a function of x representing the desired amplitude distribution of power per unit area of said reference plane, and

x_{max} is the radius of the lens.

2. The microwave lens of claim **1** wherein said lens is fabricated from a material selected from the group consisting of glass, glass-ceramic and ceramic.

3. The microwave lens of claim **1** wherein said function $F(\alpha)$ is a constant.

4. The microwave lens of claim **1** wherein said function $I(x)$ is a constant.

5. The microwave lens of claim **1** wherein said function $\phi(x)$ is a constant.

6. The microwave lens of claim **1** wherein said functions $F(\alpha)$ and $I(x)$ are constants.

7. The microwave lens of claim **1** wherein said functions $F(\alpha)$ and $\phi(x)$ are constants.

8. The microwave lens of claim **1** wherein said functions $I(x)$ and $\phi(x)$ are constants.

9. A lens for providing microwave energy having constant phase and amplitude distribution, said microwave lens being made of a dielectric material having a constant index of refraction, said lens having first and second interdependent contoured surfaces for refracting microwave energy, both of said surfaces being rotationally symmetrical about a center axis, said lens being adapted to receive energy from a microwave source, said source being located at the focal point of said lens, said first surface of said lens, which first surface is the nearest surface to said source, being defined by the parameters r, which is the distance from said focal point to said first surface and which represents the line of travel of a ray of microwave energy from said focal point to said first surface and, α, which represents the angle between r and said center axis, and the second surface thereof being defined by the parameters, x which represents the perpendicular distance from said center axis to said second surface, and y, which represents the perpendicular distance from a reference plane to said second surface, said reference plane being tangent to said second surface at said center axis, said first and second surfaces further being characterized in that said parameters r, α, x and y are determined by solving the equations

$$X = r\sin\alpha + \frac{(t(n-1)) - r(1-\cos\alpha)}{n-\cos(\alpha-\theta)}\sin(\alpha-\theta),$$

$$Y = \frac{n(r_o + t + \cos\alpha) + (r - r_o + nt)\cos(\alpha-\theta)}{n-\cos(\alpha-\theta)},$$

$$\frac{dr}{d\alpha} = \frac{rn\sin\theta}{n\cos\theta - 1},$$

and

$$r\sin\alpha + \frac{t(n-1) - r(1-\cos\alpha)}{n-\cos(\alpha-\theta)}\sin(\alpha-\theta) = X_{max}\left(\frac{\int_0^{\alpha}\sin\alpha\,d\alpha}{\int_0^{\alpha_{max}}\sin\alpha\,d\alpha}\right)^{1/2}$$

wherein

r_o is the distance along the center axis from said focal point to said first surface,

t is the distance along the center axis between said first surface and said second surface,

n is the index of refraction of the material from which the lens is made, and

θ is the angle between the line of travel of a ray of microwave energy before said ray enters said lens at said first surface, and after said ray enters said lens.

10. The microwave lens of claim **9,** wherein the initial value of the angle θ equals θ_o, and is determined by the equation

$$\frac{\sin(\alpha-\theta_o)}{n-\cos(\delta-\theta)} = \frac{\frac{X_{max}}{2r_\delta}\frac{\sin\delta}{(1-\cos\delta)} - \cos\delta - \frac{\sin\delta n\sin\theta_o}{n\cos\theta_o - 1}}{\frac{n\sin\theta_o}{n\cos\theta_o - 1}(\cos\delta - 1) - \sin\delta},$$

wherein

δ is the maximum value of the angle α,

x_{max} is the radius of the lens, and

r_δ is the value of r when α equals δ.

11. The microwave lens of claim **9,** wherein said lens is fabricated from a material selected from the group consisting of glass, glass-ceramic and ceramic.

12. The microwave lens of claim **9,** wherein one surface of said lens is substantially planar.

13. A method of making a lens for providing microwave energy having predetermined desired phase and amplitude distribution, said microwave lens being made from a dielectric material having a constant index of refraction, said lens having first and second interdependent contoured surfaces for refracting microwave energy, both of said surfaces being rotationally symmetrical about a center axis, said lens being adapted to receive energy from a microwave source, said source being located at the focal point of said lens, said first surface of said lens, which first surface is the nearest surface to said source, being defined by the parameters r, which is the distance from said focal point to said first surface and which represents the line of travel of a ray

of microwave energy from said focal point to said first surface and, α, which represents the angle between r and said center axis, and the second surface thereof being defined by the parameters, x which represents the perpendicular distance from said center axis to said second surface, and y, which represents the perpendicular distance from a reference plane to said second surface, said reference plane being tangent to said second surface at said center axis, comprising the steps of

contouring said first surface according to the equations

$dr/d\alpha = r\ n\ \sin\theta/n\cos\theta - 1$ and

$$F(x) = K_{(x_{max})} \frac{\int_0^{\alpha} F(\alpha) \sin \alpha d\alpha}{\int_0^{\alpha_{max}} F(\alpha) \sin \alpha d\alpha},$$

and contouring said second surface according to the equations

$$X = r \sin \alpha + \frac{(\varphi(x) - r_o - t - r(1 - \cos \alpha)) \sin (\alpha - \theta)}{n - \cos (\alpha - \theta)}$$

and

$$Y = \frac{n(r_o + t + \cos \alpha) - (\varphi(x) - r) \cos (\alpha - \theta)}{n - \cos (\alpha - \theta)}$$

wherein

$\phi(x)$ is a function of X representing the desired phase distribution of power at said reference plane,

$F(\alpha)$ is a function of α representing the initial power distribution from said microwave source per unit angle,

r_o is the distance along the center axis from said focal point to said first surface,

t is the distance along the center axis between said first surface and said second surface,

n is the index of refraction of the material from which the lens is made,

θ is the angle between the line of travel of a ray of microwave energy before said ray enters said lens at said first surface, and after said ray enters said lens,

$F_{(x)}$ is a function of X derived by integrating

$$\int_0^{\alpha} I(x) x dx,$$

and

$K_{(x_{max})}$ is a constant derived by integrating and evaluating

$$\int_0^{\alpha_{max}} I(x) x dx,$$

wherein

$I(x)$ is a function of X representing the desired amplitude distribution of power per unit area of said reference plane, and

x_{max} is the radius of the lens.

14. The method of claim **13** wherein said function $F(\alpha)$ is a constant.

15. The method of claim **13** wherein said function $I(x)$ is a constant.

16. The method of claim **13** wherein said function $\phi(x)$ is a constant.

17. The method of claim **13** wherein said functions $F(\alpha)$ and $I(x)$ are constants.

18. The method of claim **13** wherein said functions $F(\alpha)$ and $\phi(x)$ are constants.

19. A method of making a lens for providing microwave energy having constant phase and amplitude distribution, said microwave lens being made from a dielectric material having a constant index of refraction, said lens having first and second interdependent contoured surfaces for refracting microwave energy, both of said surfaces being rotationally symmetrical about a center axis, said lens being adapted to receive energy from a microwave source, said source being located at the focal point of said lens, said first surface of said lens, which first surface is the nearest surface to said source, being defined by the parameters r, which is the distance from said focal point to said first surface and which represents the line of travel of a ray of microwave energy from said focal point to said first surface and, α, which represents the angle between r and said center axis, and the second surface thereof being defined by the parameters, x which represents the perpendicular distance from said center axis to said second surface, and y, which represents the perpendicular distance from a reference plane to said second surface, said reference plane being tangent to said second surface at said center axis, comprising the steps of

contouring said first surface according to the equations

$dr/d\alpha = r\ n\ \sin\theta/n\cos\theta - 1$ and

$$r \sin \alpha + \left(\frac{t(n-1) - r(1 - \cos \alpha)}{n - \cos (\alpha - \theta)}\right) \sin (\alpha - \theta) = X_{max} \left(\frac{\int_0^{\alpha} \sin \alpha d\alpha}{\int_0^{\alpha_{max}} \sin \alpha d\alpha}\right)^{1/2}$$

and

contouring said second surface according to the equations

$$X = r \sin \alpha + \frac{(t(n-1) - r(1 - \cos \alpha)) \sin (\alpha - \theta)}{n - \cos (\alpha - \theta)}$$

and

$$Y = \frac{n(r_o + t + \cos \alpha) + (r - r_o + nt) \cos (\alpha - \theta)}{n - \cos (\alpha - \theta)}$$

wherein

r_o is the distance along the center axis from said focal point to said first surface,

t is the distance along the center axis between said first surface and said second surface,

n is the index of refraction of the material from which the lens is made, and

θ is the angle between the line of travel of a ray of microwave energy before said ray enters said lens at said first surface, and after said ray enters said lens.

20. The method of claim **19,** wherein the initial value of the angle θ equals θ_o, and is determined by the equation

$$\frac{\sin(\alpha-\theta_o)}{n-\cos(\delta-\theta_o)}$$

$$=\frac{\frac{X_{max}}{2r\delta}\frac{\sin\delta}{(1-\cos\delta)}-\cos\delta-\frac{\sin\delta n\sin\theta_o}{n\cos\theta_o-1}}{\frac{n\sin\theta_o}{n\cos\theta_o-1}(\cos\delta-1)-\sin\delta},$$

wherein

δ is the maximum value of angle α,
x_{max} is the radius of the lens, and
r_δ is the value of r when α equals δ.

* * * * *