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2,872,602

ARBITRARY FUNCTION GENERATORS

Filed June 13, 1955

2 Sheets-Sheet 1

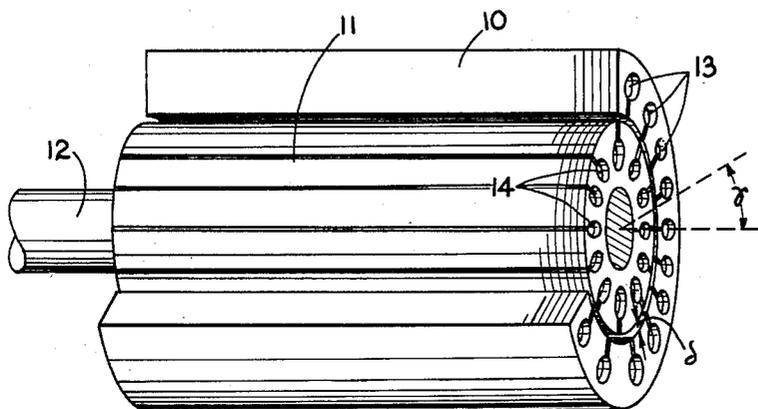


FIG. 1

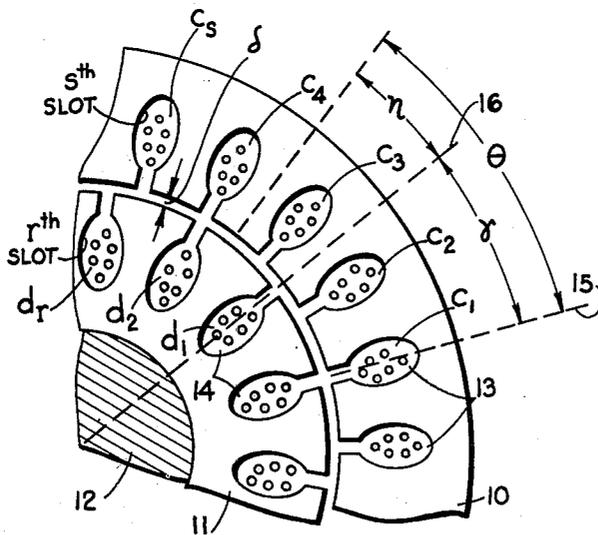


FIG. 2

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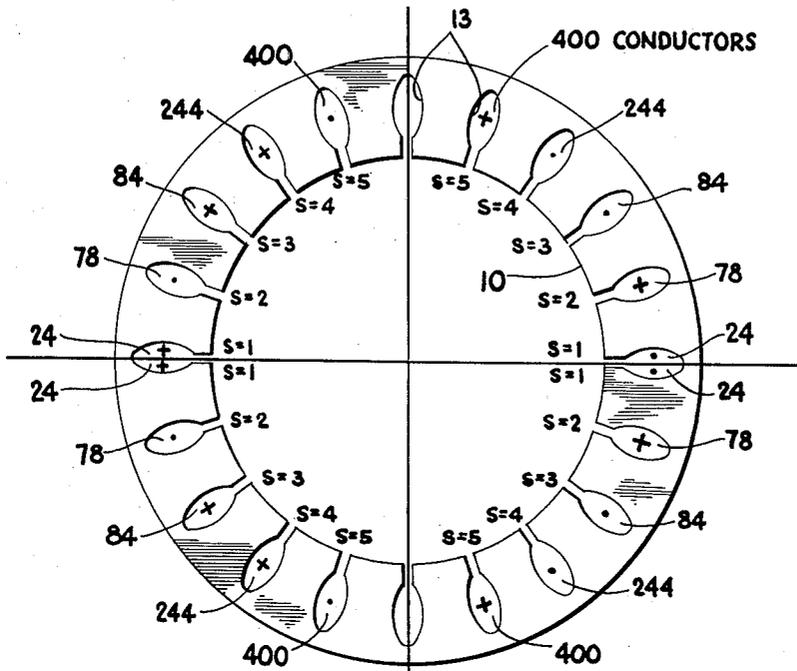


FIG. 3

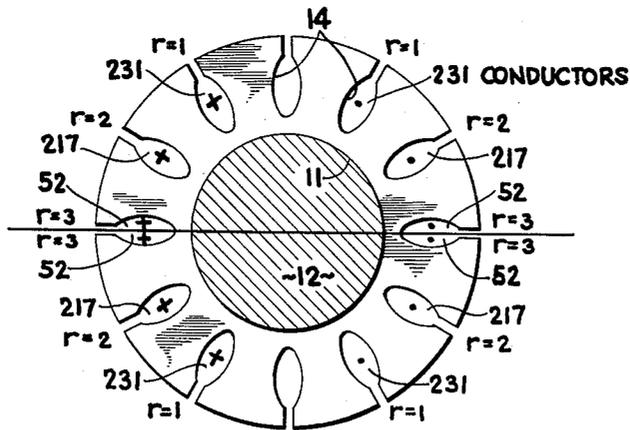


FIG. 4

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## ARBITRARY FUNCTION GENERATORS

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12 Claims. (Cl. 310-111)

This invention relates to electromagnetic induction transfer devices and more particularly, to novel transfer devices capable of synthesizing arbitrarily chosen continuous mathematical functions.

Electromagnetic voltage and torque transfer devices for generating certain specific elementary functions are well known in the art and find wide application in analog computers and fire control systems. Basically, these devices comprise a rotor body secured to a control shaft and concentrically positioned for rotation within a surrounding stator body. Both the rotor and stator are provided with discretely distributed conductor windings. By this arrangement, the degree of inductive coupling between the rotor and stator windings, when one or the other is electrically excited, is dependent upon the angular relationship of the rotor with respect to the stator.

A typical electromagnetic transfer device of the above type now in general use, is the electromagnetic resolver described and claimed in my U. S. Patent No. 2,608,682, entitled Electromagnetic Resolver, issued August 26, 1952. In the resolver, the rotor and stator are each provided with two windings mechanically 90 degrees apart. An alternating input signal applied to one of the stator windings will result in an output voltage from one of the rotor windings proportional to the product of the initial input signal and the sine or cosine of the angle of the rotor shaft or windings with respect to the stator. There is thus provided a device which may be used to continuously compute the sine or cosine of a varying angle.

The present invention has as its primary object the provision of an electromagnetic transfer device capable of generating any desired arbitrary function of the rotor shaft angle, providing such function is continuous, thereby resulting in a vastly more universal instrument than has heretofore been available.

More particularly, an object of the invention is to provide an electromagnetic transfer device which may be pre-designed in accordance with the characteristics of the desired arbitrary function to be generated, so that upon operation, the output function of the rotor shaft angle is either, (1) an electrical signal representing a synthesis of the desired arbitrary function, or (2) a restraining or opposing torque on the output shaft varying in accordance with said arbitrary function.

The basic theory underlying the present invention for attaining these and other objects and advantages, stems from the fact that any arbitrary mathematical function that is continuous over a finite period may be expanded into a Fourier series of sines and cosines. Since electromagnetic resolvers, such as described above and in the foregoing mentioned patent, are already known in the art and are individually capable of generating sine and cosine functions, it will be realized at once that a series of such resolvers individually designed to generate a sine or cosine of given amplitude and shaft angle frequency, may have their outputs combined together to provide a function that is the summation of such sines and cosines. By changing the amplitude and space frequency of each indi-

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vidual generator in accordance with the separate terms in the Fourier expansion of an arbitrary function, it is possible to provide a combined or summed output which represents the arbitrary function itself.

In accordance with the present invention, a single electromagnetic transfer device comprising a stator and rotor is provided which will immediately yield the desired arbitrary function. This result is achieved by distributing the number of conductor windings in the stator and the rotor slots in a manner determined by the coefficients of the sine and cosine terms in the Fourier series expansion of the arbitrary function. It is possible, accordingly, to provide in a single device, an output signal or shaft torque that represents any desired arbitrary function of shaft angle, including the elementary sine and cosine functions.

A better understanding of this invention will be had by referring to the accompanying drawings, in which:

Figure 1 is a schematic cut-away perspective view of an electromagnetic transfer device showing stator and rotor bodies coaxially positioned;

Figure 2 is an enlarged cross-section of a portion of the stator and rotor bodies of an electromagnetic transfer device, such as schematically shown in Figure 1;

Figure 3 is a schematic plan view of the stator illustrating the conductor distribution in accordance with an illustrative example of the invention; and,

Figure 4 is a similar schematic plan view of the rotor for the example illustrated in Figure 3.

Referring to Figure 1, there is shown a cylindrically shaped stator body 10 having a central opening within which is coaxially positioned a rotor body 11. The rotor 11 is secured to a rotor shaft 12 and is adapted to be rotated to assume different angular positions  $\gamma$  with respect to the stator. As shown, the stator is provided with a plurality of radially cut stator slots 13, and the rotor is similarly provided with a plurality of radially cut rotor slots 14. The sets of slots 13 and 14 are adapted to receive conductor windings whereby the stator body may establish an electromagnetic field across the air gap  $\delta$  to which the conductor windings of the rotor body are inductively coupled, or vice versa.

In Figure 2, there are shown symbols  $c_1, c_2, c_3, c_4, \dots$  designating the per unit conductor density in the respective stator slots 13, and symbols  $d_1, d_2, \dots$  designating the per unit conductor density in the respective rotor slots 14. Let  $c_s$  equal the per unit conductor density in the  $s^{\text{th}}$  stator slot, and  $d_r$  equal the per unit conductor density in the  $r^{\text{th}}$  rotor slot. The actual number of conductors in the  $s^{\text{th}}$  and  $r^{\text{th}}$  slots is determined by multiplying the absolute values of  $c_s$  and  $d_r$  by the total number of conductors employed in the stator and the rotor respectively.

Stated another way, the per unit conductor density,  $c_s$  in the  $s^{\text{th}}$  stator slot, is, in magnitude, the ratio of the actual number of physical conductors in the  $s^{\text{th}}$  stator slot to the total number of net effective conductors in all the stator slots comprising the particular stator distributed winding made up of a group of  $c_s$ 's, with a plus sign prefixed for one chosen (reference) direction of actual current flow in the conductors in the slots, and with a minus sign prefixed for the opposite direction of actual current flow in the conductors in slots having such current flow opposite to the reference direction. The same situation obtains for the  $d_r$  per unit conductor density in the  $r^{\text{th}}$  rotor slot. The factor 100 times the "per unit conductor density" is the percent conductor density in a particular slot.

In accordance with the present invention, the conductor winding distribution among the slots of the stator and rotor; that is, the actual values of the  $c_s$ 's, and  $d_r$ 's, are uniquely predetermined by the coefficients of the Fourier Series expansion of the arbitrary function to be generated.

With such distribution, the variation of the ratio of the output signal to the input signal or the variation of the rotor shaft torque with changes in the rotor shaft angle may be made to represent the desired arbitrary function.

The manner in which  $c_s$  and  $d_r$  are uniquely related to the coefficients of the Fourier series in order to determine the distribution of the conductors in the slots will now be described.

Referring to Figure 2, let the dashed line 15 represent a reference angle line for the stator 10. Any point in the airgap  $\delta$  having a certain flux density generated by the stator windings alone may then be located by specifying an angle  $\theta$  so many degrees from the reference line 15.

Similarly, let the dashed line 16 represent a reference angle line for the rotor. Any point in the airgap  $\delta$  having a certain flux density generated by the rotor windings alone, may then be located by specifying an angle  $\eta$  so many degrees from the reference line 16.

The angle of shaft rotation  $\gamma$  representing the degree of rotation of the rotor with respect to the stator, will then be the angle between the reference lines 15 and 16 as shown in Figure 2.

The arbitrary function to be generated in accordance with the invention will be represented by changes in the output of the device with variations of  $\gamma$ ; therefore, let  $f(\gamma)$  represent the desired arbitrary function.

In the case where it is desired to synthesize the arbitrary function as an electrical output signal,

$$f(\gamma) = \frac{E_o}{E_i}(\gamma) \quad (1)$$

where

$E_o$  = output voltage from the device, and  
 $E_i$  = input voltage to the device.

In the case where it is desired to provide a shaft torque which varies in accordance with an arbitrary function, this torque will be proportional to the product of applied stator and rotor voltage, as well as a function of the shaft angle, or:

$$T(\gamma) = T(\gamma, E_r, E_s) \quad (2)$$

where

$T(\gamma)$  represents the arbitrary torque function,  
 $E_r$  = voltage applied to the rotor, and  
 $E_s$  = voltage applied to the stator.

Consider first the case of a voltage transfer arbitrary function generator for generating an electrical signal as specified in (1). In accordance with Fourier's theorem,  $f(\gamma)$  may be expanded into a series of sines and cosines. Consider first an arbitrary function of the type that may be represented by a series of cosines alone, thus:

$$f(\gamma) = \sum_n A_n \cos n\gamma \quad (3)$$

where  $n$  = harmonic order, and  $A_n$  are the Fourier coefficients. For these cases:

Let  $f(\theta)$  equal the normalized stator distribution of flux density generated in the airgap  $\delta$  for different angles of position  $\theta$  around the stator periphery, referred to the stator reference line 15, when the stator conductor distribution alone is excited.

Let  $f(\eta)$  equal the normalized rotor distribution of flux density generated in the airgap  $\delta$  for different angles of position  $\eta$  around the rotor periphery, referred to the rotor reference line 16, when the rotor conductor distribution alone is excited.

$f(\theta)$  and  $f(\eta)$  may then be represented by Fourier's Series as follows:

$$f(\theta) = \sqrt{2} \sum_n \frac{A_{ns}}{n} \sin n\theta \quad (4)$$

$$f(\eta) = \sqrt{2} \sum_n \frac{A_{nr}}{n} \sin n\eta \quad (5)$$

where

$$\sqrt{2} \frac{A_{ns}}{n} \text{ and } \sqrt{2} \frac{A_{nr}}{n}$$

equal the  $n^{\text{th}}$  harmonic amplitude for the stator and rotor respectively.

From Figure 2 it will be noted that:

$$\gamma = \theta - \eta \quad (6)$$

Now, the convolution integral of  $f(\theta)$  and  $f(\eta)$  for  $\gamma = \theta - \eta$  is given, in general, by

$$\int_0^{2\pi} f(\theta) \cdot f(\eta) d\theta$$

Upon the substitution of  $\theta - \gamma$  for  $\eta$ , from Equation 6, and  $f(\theta)$  and  $f(\eta)$  as given in Equations 4 and 5, into this convolution integral and integrating, there results:

$$\int_0^{2\pi} \sum_n \frac{A_{ns}}{n} \sin n\theta \cdot \sqrt{2} \sum_n \frac{A_{nr}}{n} \sin n(\theta - \gamma) d\theta = \sum_n \frac{A_{nr}}{n} \cdot \frac{A_{nr}}{n} \cos n\gamma \quad (7)$$

Comparing the right side of Equation 7 with the right side of Equation 3, it will be seen that the functions are identical term by term, and hence in their entirety, if the coefficients are made equal, thus:

$$\frac{A_{ns}}{n} \cdot \frac{A_{nr}}{n} = A_n \quad (8)$$

whereby the arbitrary function,  $f(\gamma)$  of Equation 3 is given by

$$f(\gamma) = \sum_n \frac{A_{ns}}{n} \cdot \frac{A_{nr}}{n} \cos n\gamma \quad (9)$$

The next step is to relate the quantities  $A_{ns}$  and  $A_{nr}$  to the per unit conductor density in all stator and rotor slots as defined generally by  $C_s$  and  $d_r$  for the  $s^{\text{th}}$  and  $r^{\text{th}}$  slots respectively.

Referring again to Figure 2, assume the stator has a total of " $p$ " slots distributed over  $2\pi$  radians in  $\theta$  and that the rotor has a total of " $q$ " slots distributed over  $2\pi$  radians in  $\eta$ . The per unit slot-conductor densities  $c_1, c_2, c_3, c_4, \dots, c_s, \dots, c_p$ , where  $c_p$  represents the per unit slot-conductor density in the last slot of the " $p$ " stator slots, are distributed in these " $p$ " slots respectively, adjacent slots being separated  $2\pi/p$  radians. Similarly, the per unit slot-conductor densities  $d_1, d_2, \dots, d_r, \dots, d_q$ , where  $d_q$  represents the per unit slot-conductor density in the last of the " $q$ " rotor slots, are distributed in these " $q$ " slots, respectively, adjacent slots being separated by  $2\pi/q$  radians.

In the case of continuous windings on both the stator and rotor, two conditions must be met:

$$\sum_1^p c_s = 0 \quad (10)$$

and,

$$\sum_1^q d_r = 0 \quad (11)$$

where each  $c_s$  and  $d_r$  is given a positive or negative sign in terms of a convenient arbitrary current carrying direction reference. Conditions 10 and 11 are necessary, since magnetic flux lines must close on themselves. Under these conditions, it will be immediately evident that a maximum of  $(p-1)$  independent variables exist for the stator slot conductor density distribution, and a maximum of  $(q-1)$  independent variables exist for the rotor slot density distribution.

Additional constraints further reducing the number of

Independent variables are imposed for each occurrence of an axis of symmetry or asymmetry in the distributed ordered array of slot-conductor densities. Such additional constraints vary in their form dependent upon the location and spacing of the symmetrical or asymmetrical axes.

Generally, in terms of the total number of equi-spaced slots ( $p$  or  $q$ ), the generalized slot-conductor densities ( $c_s$  or  $d_r$ ), and the total numbers and locations of the axes of symmetry and asymmetry in the distributed ordered arrays of slot-conductor densities, the following expressions apply:

$$\alpha_n + \sum_{s=1}^{s=p} c_s g_s(s, n, p) = A_{ns} \quad (12)$$

$$\beta_n + \sum_{r=1}^{r=q} d_r g_R(r, n, q) = A_{nr} \quad (13)$$

in which:

$\alpha_n$  is a known constant for a given  $p$  and  $n$  and for known axes and locations of axes of symmetry and asymmetry of the stator slot-conductor density distributed array, and varies at most, only with  $n$ ;

$\beta_n$  is a known constant for a given  $q$  and  $n$  and for known axes and locations of axes of symmetry and asymmetry of the rotor slot-conductor density distributed array, and varies at most, only with  $n$ ;

$c_s$  is, as defined previously, the per unit stator slot-conductor density, with algebraic sign, in the  $s^{\text{th}}$  stator slot;

$d_r$  is, as defined previously, the per unit rotor slot-conductor density, with algebraic sign, in the  $r^{\text{th}}$  rotor slot;

$g_s(s, n, p)$  denotes the "stator generating function" whose form is known and determined by the known axes and locations of axes of symmetry and asymmetry of the stator slot-conductor distributed ordered array over  $2\pi$  radians in  $\theta$ ; and which is a function of the slot number " $s$ " on the stator, of the order " $n$ " of the space harmonic referred to  $2\pi$  radians in  $\theta$  as the fundamental or first harmonic, and of " $p$ ," the number of equi-spaced slots in the stator; and,

$g_R(r, n, q)$  denotes the "rotor generating function" whose form is known and determined by the known axes and locations of axes of symmetry and asymmetry of the rotor slot-conductor distributed ordered array over  $2\pi$  radians in  $\eta$ , and which is a function of the slot number " $r$ " on the rotor, of the order " $n$ " of the space harmonic referred to  $2\pi$  radians in  $\eta$  as the fundamental or first harmonic, and of " $q$ ," the number of equi-spaced slots in the rotor.

For known " $p$ " and " $q$ " and for known axes and locations of axes of symmetry and asymmetry of the stator and rotor slot-conductor density distributed arrays, Equations 12 and 13 yield a set of simultaneous equations as follows:

$$\left[ \alpha_1 + \sum_{s=1}^{s=p} c_s g_s(s, 1, p) \right] \left[ \beta_1 + \sum_{r=1}^{r=q} d_r g_R(r, 1, q) \right] = A_{1s} \cdot A_{1r} = A_{11}$$

$$\left[ \alpha_n + \sum_{s=1}^{s=p} c_s g_s(s, n, p) \right] \left[ \beta_n + \sum_{r=1}^{r=q} d_r g_R(r, n, q) \right] = A_{ns} \cdot A_{nr} = n^2 A_n$$

The original Fourier analysis of the arbitrary function to be synthesized is carried out to a sufficiently high order,  $n$  of harmonic amplitudes,  $A_n$ , to approximate the

given function to within prescribed tolerances over a prescribed range of the independent variable  $\gamma$ . These  $A_n$ 's then, are given knowns in the set of simultaneous Equations 14. Further, the  $\alpha_n$ 's,  $\beta_n$ 's,  $g_s$ 's and  $g_R$ 's are known. Only the  $c_s$ 's and  $d_r$ 's are unknowns in the set of Equations 14 and therefore these equations may be simultaneously solved to provide the values of the  $c_s$ 's and  $d_r$ 's, there being as many equations as there are unknowns.

In other words, if the number of the highest harmonic amplitude,  $A_n$  (not the order), is denoted by  $N$ , and the least number of independent  $c_s$ 's and  $d_r$ 's to synthesize the arbitrary function required, are denoted by  $S$  and  $R$  respectively, then:

$$N = S + R \quad (15)$$

and therefore, it is possible to solve for each  $c_1, c_2, \dots$  and each  $d_1, d_2, \dots$  from Equations 14.

Conductor windings are then applied to the stator and rotor slots 13 and 14 in accordance with the solutions for  $c_s$  and  $d_r$ . When an input signal is then applied to the device to excite the particular distributed winding array, the electrical output signal will vary with the shaft angle  $\gamma$  in accordance with the initially selected arbitrary function  $f(\gamma)$ .

It will be recalled that the above analysis is for generating an arbitrary function which may be expressed by a Fourier series of cosines, Equation 3. Another class of arbitrary functions may be represented by a Fourier series of sines, thus:

$$f(\gamma) = \sum_n B_n \sin n\gamma \quad (16)$$

The analysis for relating  $c_s$  and  $d_r$  to the Fourier coefficients,  $B_n$ , in Equation 16, is carried out in an identical manner to that given above for Equation 3, except for a rotation of  $\pi/2$  radians of the  $\theta$  and  $\eta$  reference lines.

In the most general case in which the arbitrary function is represented by both summations of the Fourier series, thus:

$$f(\gamma) = \sum_n (A_n \cos n\gamma + B_n \sin n\gamma) \quad (17)$$

the  $c_s$ 's and  $d_r$ 's determined by the  $B_n$ 's, may be superimposed on the  $c_s$ 's and  $d_r$ 's determined by the  $A_n$ 's, resulting in a single, composite stator slot-conductor density distribution on the stator, and in a single, composite rotor slot-conductor density distribution on the rotor.

It will thus be seen that the conductor windings in the different slots are provided with weighted operating characteristics from the standpoint of the number of conductor windings in each slot and that such weighted operating characteristics are in accordance with the solution of a plurality of simultaneous equations. Each simultaneous equation involves the weighted operating characteristics of a different harmonic generated by the windings in a pair of Fourier Series. The product of the pair of Fourier Series expresses the arbitrary func-

tion to be generated with progressive displacements between the rotor and the stator.

Furthermore, each Fourier Series in the pair individ-

(14)

ually express the arbitrary functions to be generated with progressive relative orientations along the first member or along the second member.

Considering now the case of a torque transfer arbitrary function generator as described by Equation 2, the restraining or opposing torque function is given by:

$$T(\gamma) = I_s \cdot I_r \frac{\partial L_m(\gamma)}{\partial \gamma} \quad (18)$$

where  $L_m(\gamma)$  ideally denotes the coefficient of mutual induction between the stator and rotor as a function of the shaft angle  $\gamma$ , and  $I_s$  and  $I_r$  respectively are the current flowing in the stator and rotor windings as a result of the applied voltages  $E_s$  and  $E_r$  of Equation 2.

Assume first, that the arbitrary torque function is expressible by a Fourier Series of sines, thus:

$$T(\gamma) = \sum_n A'_n \sin n\gamma \quad (19)$$

where  $A'_n$  denote the Fourier coefficients of the torque function.

With respect to the  $A'_{ns}$  and  $A'_{nr}$  of Equations 4 and 5 of the voltage transfer device, set

$$A'_{ns} = A'_{ns} I_s \quad (20)$$

and

$$A'_{nr} = A'_{nr} I_r \quad (21)$$

then:

$$F(\theta) = \sqrt{2} I_s \sum_n \frac{A'_{ns}}{n} \sin n\theta \quad (22)$$

$$F(\eta) = \sqrt{2} I_r \sum_n \frac{A'_{nr}}{n} \sin n\eta \quad (23)$$

in which  $F(\theta)$  and  $F(\eta)$  are respectively proportional to the stator and rotor flux density distributions in the air gap, each acting independently of the other, and in which the currents  $I_s$  and  $I_r$  are placed in evidence.

Referring to Equation 18,  $I_s \cdot I_r \cdot L_m(\gamma)$  is proportional to the convolution integral of  $F(\theta)$  and  $F(\eta)$ , thus:

$$I_s \cdot I_r \cdot L_m(\gamma) = K \int_0^{2\pi} F(\theta) \cdot F(\eta) d\theta \quad (24)$$

where  $K$  is the proportionality constant and is independent of  $\gamma$ ,  $I_s$ , and  $I_r$ . The arbitrary torque function of  $\gamma$  given by Equation 18, is then given by:

$$T(\gamma) = K \frac{\partial}{\partial \gamma} \left[ \int_0^{2\pi} F(\theta) \cdot F(\eta) d\theta \right] \quad (25)$$

in which again,  $\gamma = \theta - \eta$ , resulting in:

$$T(\gamma) = K I_s \cdot I_r \sum_m \frac{A'_{ns} \cdot A'_{nr}}{m} \sin m\gamma \quad (26)$$

As in the case of the voltage transfer arbitrary function generator analysis, Equations 19 and 26 can be equated term by term, by making:

$$[K I_s \cdot I_r] \frac{A'_{ns} \cdot A'_{nr}}{n} = A'_n \quad (27)$$

By dropping the primes in Equation 27, it is evident that a voltage transfer device whose normalized harmonic amplitudes are given by:

$$\frac{A_{ns} \cdot A_{nr}}{n}$$

is also a torque transfer device whose normalized harmonic amplitudes are given by

$$\frac{A_{ns} \cdot A_{nr}}{n}$$

and whose reference lines for  $\gamma$  in the output function are space phase displaced  $\pi/2$  radians with respect to the reference lines of the voltage transfer device (lines 15 and 16 in Figure 2).

To determine  $c_s$  and  $d_r$  for the torque transfer device, a set of simultaneous equations whose left hand sides are identical to those in the set of Equations 14, but whose right hand sides in the general term, are written  $nA_n$  instead of  $n^2A_n$ , are used. Thus, the arbitrary torque functions may be reproduced by distributing the stator and rotor conductor windings in accordance with the new  $c_s$ 's and  $d_r$ 's.

The analysis for relating  $c_s$  and  $d_r$  to the Fourier coefficients for a torque function expressible by:

$$T(\gamma) = \sum_n B'_n \cos n\gamma \quad (28)$$

is executed in an identical manner, except for a rotation of  $\pi/2$  radians in the  $\theta$  and  $\eta$  reference lines.

In the most general case for the torque function, wherein,

$$T(\gamma) = \sum_n (A'_n \sin n\gamma + B'_n \cos n\gamma) \quad (29)$$

the  $c_s$ 's and  $d_r$ 's determined by the  $B'_n$ 's, may be superimposed on the  $c_s$ 's and  $d_r$ 's determined by the  $A'_n$ 's, as in the case of the voltage transfer device, resulting in a single composite  $c_s$  stator slot-conductor density distribution on the stator and in a single composite  $d_r$  rotor slot-conductor distribution on the rotor.

Summarizing, the preceding formulas elicit the unique stator and rotor per unit slot conductor density distributions over all slots for the generation of the given arbitrary empirical or analytical function as a voltage or torque transfer function within the limitations of the total numbers of stator and rotor slots. Thus, having once determined the stator and rotor per unit slot conductor densities, they may be respectively multiplied by the total number of net effective stator conductors and the total number of net effective rotor conductors to yield the actual number of conductors in each slot of the stator and rotor and the relative directions of current flow therein.

The total number of actual conductors and the wire size employed on any stator or rotor winding in slots of a given area is determined, among other factors, by the total slot volume, the slot fill factor, insulation thickness, ampere turns for proper non-saturating use of the magnetic material, air gap width, maximum number of conductors in the maximum filled slot, transformation ratio, total winding impedance, and the winding "Q" requirements in the intended application.

As a specific example among many which have been reduced to actual practice in accordance with the instant invention, consider the application of Equation 16:

$$f(\gamma) = \sum_n B_n \sin n\gamma$$

for use in connection with functions having sine symmetry about the origin,  $\gamma=0$ . In the example under consideration, assume the arbitrary function to be generated is expressed by:

$$f(\gamma) = \frac{1 - \cos \gamma}{\gamma}$$

This function is very useful in modern computers and guided missile control systems and is an example of the latter above mentioned type of function having sine symmetry about the origin,  $\gamma=0$ .

Assume that it is desired to develop this function in a single arbitrary function generator over the values  $-0.636620 \leq f(\gamma) \leq +0.636620$  whereby one servomechanism and one resolver or potentiometer in a conventional computer network may be eliminated. The corresponding range in  $\gamma$  is

$$-\frac{\pi}{2} \leq \gamma \leq +\frac{\pi}{2}$$

In order to develop the  $B_n$  harmonic amplitudes of the given function which step is a prerequisite to the unique

design of the conductor distributions on the stator and rotor, the customary Fourier analysis is performed with  $\gamma$  values taken in five degree intervals from zero degrees to ninety degrees. In the following table, "i" denotes the ordered numbering of these intervals, " $f_i$ " the corresponding function values, and " $f_{in}$ " denotes the corresponding function values normalized to unity:

i	$f_i$	$f_{in}$
0	0.000000	0.000000
1	0.043602	0.068490
2	0.087044	0.136728
3	0.130153	0.204444
4	0.172767	0.271382
5	0.214726	0.337291
6	0.255873	0.401924
7	0.296052	0.465037
8	0.335117	0.526400
9	0.372923	0.585786
10	0.409335	0.642982
11	0.444224	0.697785
12	0.477465	0.750000
13	0.508947	0.799452
14	0.538564	0.845974
15	0.566220	0.889416
16	0.591831	0.929046
17	0.615319	0.965540
18	0.636620	1.000000

The usual Fourier analysis yields the following harmonic amplitudes through the 35th order,  $n=35$ . Since the function has sine symmetry, only the odd harmonics in the Fourier expansion need be computed:

n	$B_n$
1	0.889881
3	-0.053638
5	0.019044
7	-0.008804
9	0.006037
11	-0.004142
13	0.003056
15	-0.002380
17	0.001931
19	-0.001622
21	0.001400
23	-0.001239
25	0.001119
27	-0.001032
29	0.000969
31	-0.000925
33	0.000896
35	-0.000883

The immediate problem is to generate exactly in amplitude and phase the maximum number of these harmonics

possible as limited only by the number of stator and rotor slots available and the physical constraints of electromagnetic induction fields.

Since it has been initially recognized and noted that the function to be synthesized has sine symmetry about the origin,  $\gamma=0$ , the stator and rotor step-wise flux density distributions (mmf functions) are chosen from that class which also has this same sine symmetry but not sinusoidal shape. Thus, letting  $p$  equal the number of stator slots and  $q$  equal the number of rotor slots, the general Equations 12 and 13 for this class of functions, respectively take the forms:

$$\sin \frac{n\pi}{p} \sum_s a_s \sin \frac{h(s)n\pi}{p} = B_{ns} \quad (30)$$

$$(-1)^{\frac{n-1}{2}} \frac{\sin \frac{Kn\pi}{q}}{Kn\pi} \sum_r d_r \sin \frac{m(r)n\pi}{q} = B_{nr} \quad (31)$$

The K factor in Equation 31 is the so-called per unit skewness and constitutes that fraction of one slot-to-slot width that the rotor stack skews in going from the front end of the rotor body to the rear end of the body;  $a_s$  is defined by  $c_s = a_s - a_{s-1}$ ;  $h(s)$  is a distribution function over one quadrant of the stator slots which is repeated over the other three quadrants;  $m(r)$  is another distribution function over one quadrant of the rotor slots which is repeated over the other three quadrants;  $s$ , as before, is the sequential numbering of the stator slots in one quadrant; and  $r$  is the sequential numbering of the rotor slots in one quadrant. Specifically, for the case of  $p=20$  stator slots and  $q=12$  rotor slots, the above equations become explicitly:

$$\sin \frac{n\pi}{20} \sum_{s=1}^{s=5} a_s \sin \frac{(2s-1)n\pi}{20} = B_{ns} \quad (32)$$

$$(-1)^{\frac{n-1}{2}} \frac{\sin \frac{Kn\pi}{12}}{Kn\pi} \sum_{r=2}^{r=3} d_r \sin \frac{2r n\pi}{12} = B_{nr} \quad (33)$$

The general Equations 14 for sine symmetry are then expressed by  $B_{ns} \cdot B_{nr} = n^2 B_n$  and may be written out:

$$(34) \left\{ \begin{aligned} & \left[ \sin \frac{\pi}{20} \sum_{s=1}^{s=5} a_s \sin \frac{(2s-1)\pi}{20} \right] \left[ \frac{\sin \frac{K\pi}{12}}{K\pi} \sum_{r=1}^{r=3} d_r \sin \frac{r\pi}{6} \right] = B_1 \\ & \left[ \sin \frac{3\pi}{20} \sum_{s=1}^{s=5} a_s \sin \frac{(2s-1)3\pi}{20} \right] \left[ \frac{\sin \frac{K\pi}{4}}{K\pi} \sum_{r=1}^{r=3} d_r \sin \frac{r\pi}{2} \right] = 9B_3 \\ & \left[ \sin \frac{5\pi}{20} \sum_{s=1}^{s=5} a_s \sin \frac{(2s-1)5\pi}{20} \right] \left[ \frac{\sin \frac{K5\pi}{12}}{K5\pi} \sum_{r=1}^{r=3} d_r \sin \frac{r5\pi}{6} \right] = 25B_5 \end{aligned} \right.$$

Because the  $c_s$ 's (as expressed in terms of  $a_s$ ) and  $d_r$ 's are normalized to unity:

$$|a_s| = 1, \sum_{s=1}^{s=5} c_s = 1 \tag{35}$$

and,

$$\sum_{r=1}^{r=3} d_r = 1 \tag{36}$$

The above system of equations has four independent  $a_s$ 's (or  $c_s$ 's), two independent  $d_r$ 's, and one independent K. It is, therefore, possible to solve for them from the above system of equations by utilizing the known  $B_1, B_3, B_5, B_7, B_9, B_{11}, B_{13},$  and  $B_{15}$  of the given function set forth in the previous table and employing the well known theory of solution of simultaneous equations. The results are tabulated as follows:

	$a_s$	
$a_1$	0.192847	$c_s = a_s - a_{s-1}$  $K = 0.890003$
$a_2$	-0.424190	
$a_3$	0.251114	
$a_4$	2.198569	
$a_5$	-1.000000	
$c_s$		
$c_1$	0.192847	
$c_2$	-0.617037	
$c_3$	0.765304	
$c_4$	1.947455	
$c_5$	-3.198569	
$\sum c_s$	1.000000	
$d_r$		
$d_1$	0.462676	
$d^2$	0.433387	
$d^3$	0.103937	
$\sum d_r$	1.000000	

In general, for this case of  $p=20$  stator slots and  $q=12$  rotor slots, the generated  $B_n$  obtained by substituting the above K,  $d_r$  and  $c_s$  values back in to Equations 34 will be exactly equal to the original  $B_n$  calculated previously for the given function for  $n=1$  to 35 only over the order of  $n$  from 1 to 15 inclusive utilized to obtain the K,  $d_r$  and  $c_s$  values. However, they are the unique rotor skewness and rotor and stator conductor distributions for the generation of the given function,

$$\frac{1 - \cos \gamma}{\gamma}$$

over the specified range, for the case of  $p=20$  stator slots and  $q=12$  rotor slots with maximum harmonic matching for this slot combination.

A comparison between the calculated  $B_n$  of the given function and the generated  $B_n$  of the function-generator is set forth below:

$n$	$B_n$ Calculated	$B_n$ Generated
1	0.889881	0.889900
3	-0.053638	-0.053644
5	0.019044	0.019043
7	-0.009804	-0.009804
9	0.008037	0.008036
11	-0.004142	-0.004141
13	0.003056	0.003056
15	-0.002380	-0.002380
17	0.001931	0.000087
19	-0.001522	-0.000030
21	0.001400	0.000207
23	-0.001239	-0.000292
25	0.001119	0.001301
27	-0.001032	-0.000012
29	0.000969	0.000011
31	-0.000925	-0.000015
33	0.000896	0.000828
35	-0.000883	-0.001024

This table establishes the validity of the statement that the harmonics of the original Fourier Series of the function and the harmonics of the generated function match exactly, and exactly only, up to and through  $n=15$ , the maximum possible and unique matching within the limits of the slot numbers,  $p=20, q=12$ .

It is desirable to finally compare the normalized function values developed by the function generator, with the normalized values of the given function. This is done, utilizing the first 35 harmonics, those above that having dropped to very small values and not contributing substantially to the error percentage.

$i$	degrees	$f_{in}$	$f_{in}$ generated	Percent Error
0	0	0.000000	0.000000	0.0000
1	5	0.068490	0.068823	0.0333
2	10	0.136728	0.134675	-0.2053
3	15	0.204444	0.206472	0.2028
4	20	0.271382	0.271708	0.0326
5	25	0.337291	0.336852	-0.0439
6	30	0.401924	0.400151	-0.1773
7	35	0.465037	0.467209	0.2172
8	40	0.526400	0.525364	-0.1036
9	45	0.585786	0.587962	0.2176
10	50	0.642982	0.640068	-0.2914
11	55	0.697755	0.697463	-0.0322
12	60	0.750000	0.752416	0.2416
13	65	0.799432	0.800814	0.1362
14	70	0.845974	0.842186	-0.3788
15	75	0.889415	0.889085	-0.0361
16	80	0.929946	0.931874	0.2228
17	85	0.95840	0.971470	0.4930
18	90	1.000000	0.959623	-1.0377

Nowhere except in the last interval does the percent error exceed 0.5 percent for this rather unusual heretofore unavailable, practical function, and the R. M. S. error is less than 0.08%. Since even this remaining error is quite periodic, it may be still further minimized by a simple auxiliary winding designed on the above principles for the main function.

The configuration for the rotor per-unit slot conductor densities demonstrates a ratio of 4.451504 between the maximum loaded slot ( $r=1$ ) and the least loaded slot ( $r=3$ ). On the stator, the situation is somewhat different, because some of the slots in a quadrant have conductors which are oppositely poled (reversed current direction) to others in the same quadrant of slots. Since summation of the absolute magnitudes of  $c_s$  is the base reference for the total number of physical conductors in one quadrant of the stator slots, and equals 6.631212, whereas the absolute magnitude of the summation of  $c_s$  is always equal to unity, this winding requires a total number of physical conductors equal to 6.631212 times the total number of net effective conductors. The ratio between the maximum loaded slot ( $s=5$ ) and the least loaded slot ( $s=1$ ) is 16.586044.

Calculation of the required self-inductance of the stator winding in a practical size 23 case, for an air-gap  $\delta$  of 0.00554 inch and for an existing stator slot area of 0.02355 square inch necessitates a total net effective number of conductors per stator winding of 500. This is 125 net effective conductors per quadrant of slots, but a total of 830 physical conductors per quadrant of stator slots of which

$$\frac{3.198569}{6.631212} \times 830 = 400$$

are in the maximum loaded slot,  $s=5$ . Postulating 60% fill factor, including slot insulation, wire insulation, lay factor, and so forth, the copper area of 400 conductors in the slot  $s=5$  equals 0.014130 square inch. Hence, a wire size for which the copper diameter is 0.0126 inch, is specified, for example, AWG No. 28.

Referring now to Figure 3, the physical stator conductor distribution in the stator slots is shown by the numerals which indicate the actual physical number of conductors in each slot. The dots indicate that the direction of current flow in certain slots is coming out of the

drawing, while the crosses indicate that the direction of current flow in certain slots is passing into the drawing. The actual windings may be toroidal, latitude, or effected in any other convenient manner. The step-wise air-gap flux density distribution (mmf. function) generated by this winding over 360° of the inside periphery of the stator, directly corresponds to the step-wise poled conductor distribution given in Figure 3.

Calculation of the required self-inductance of the rotor winding in a practical size 23 case, for an air-gap  $\delta$  of 0.00554 inch and for an existing rotor slot area of .02461 square inch necessitates a total net effective number of conductors per rotor winding of 2000. This number corresponds to the actual total number of physical conductors in one quadrant, since the direction of current flow in any one quadrant is uniform. Thus, there are 500 conductors per rotor quadrant.

The corresponding poled conductor distribution is illustrated in Figure 4 wherein a 60% fill factor is used and No. 28 AWG copper wire is employed. As in the case of Figure 3, the numerals indicate the actual physical number of conductors in each slot and the dots and crosses indicate the corresponding direction of current flow.

The above specific example thus illustrates the manner in which the slot conductor density distributions for the stator and rotor are determined in accordance with the characteristics of the arbitrary function

$$f(\gamma) = \frac{1 - \cos \gamma}{\gamma}$$

whereby the output of the generator represents the arbitrary function.

It will be seen from the foregoing description that the present invention provides an electromagnetic transfer device which is far more versatile than any heretofore known in the art, in that it may be employed to generate any arbitrary continuous function, reproducing such function as either an electrical voltage signal or as a shaft torque.

What is claimed is:

1. An arbitrary function generator comprising: a first body; a second body co-axially positioned in concentric relationship to said first body for rotation relative to said first body through a variable angle  $\gamma$  measured from a radial reference line on said first body, said second body and first body defining an annular air gap therebetween; said first body having a plurality of discrete radially cut slots circumferentially spaced about its axis; electrical conductors positioned in said slots; each slot, in sequential order from said reference line containing a discrete number of said electrical conductors such that the ratio of the number of conductors in any one slot to the total number of conductors in all of said slots defines a unique absolute per unit conductor density in said any one slot; means for passing current through said conductors in specified directions to establish a first body slot step-wise flux density distribution in the adjacent air-gap varying over different circumferential points measured from said reference line; and conductor means on said second body adapted to be inductively coupled to said first body step-wise slot flux density distribution upon rotation of said second body through various values of the angle  $\gamma$ , the effective per unit slot conductor densities in said first body slots as derived from the absolute per unit conductor densities and said specified directions of current flow, having values such as to provide said step-wise flux density distribution with harmonic amplitudes and signs equal to the term by term co-efficients of the Fourier Series expansion of said arbitrary function, whereby the signal induced in said conductor means of said second body represents a function of said angle  $\gamma$  corresponding to said arbitrary function.

2. An arbitrary function generator comprising: a stator body; a rotor body co-axially positioned in said stator body for rotation relative to said stator body through a variable angle  $\gamma$  measured from a radial reference line

on said stator, said rotor and stator defining an annular air-gap therebetween; said rotor and stator bodies each having a plurality of discrete radially cut slots circumferentially spaced about their respective axes; electrical conductors positioned in said rotor and stator slots; each slot in said rotor and stator containing a discrete number of said electrical conductors such that the ratio of the number of conductors in any one rotor slot to the total number of conductors in all of said rotor slots defines the absolute per unit conductor density in said any one rotor slot and the ratio of the number of conductors in any one stator slot to the total number of conductors in all of said stator slots defines the absolute per unit conductor density in said any one stator slot; and means for passing current through said conductors in each of said rotor and stator slots in specified directions to establish a rotor and stator slot step-wise flux density distribution in the adjacent air-gap varying over different circumferential points measured from said reference line; the respective electrical conductors associated with said rotor and stator being inductively coupled upon rotation of said rotor through various values of the angle  $\gamma$ , the products of a first function of the effective per unit conductor density distribution in each of said rotor slots with a second function of the effective per unit conductor density distribution in each of said stator slots, as derived from said rotor and stator absolute per unit conductor densities and said rotor and stator slot specified directions of current flow, being term by term respectively proportional to and of the same sign as the term by term coefficients of the Fourier Series expansion of said arbitrary function, whereby the output signal of said generator represents a function of said angle  $\gamma$  corresponding to said arbitrary function.

3. In apparatus for generating any desired arbitrary function having only a single value at each position along one of a pair of transverse axes, a first member, a second member movable relative to the first member, a first plurality of generating means disposed at spaced positions along the first member and a second plurality of generating means disposed at spaced positions along the second member, the generating means in the first plurality being interrelated and being provided with characteristics to generate, upon progressive displacements between the first and second members, the desired arbitrary function as indicated by a convolution product integral which represents the integral of the product of a pair of functions, one representing the characteristics of the signals generated by the first generating means upon progressive relative orientations of the first member and the other representing the characteristics of the signals generated by the second generating means upon progressive relative orientations of the second member.

4. In apparatus for generating any desired arbitrary function having only a single value at each position along one of a pair of transverse axes, a first member provided with a plurality of apertures at spaced positions along one periphery, a second member provided with a plurality of apertures at spaced positions along the periphery facing the apertures in the first member, a first plurality of generating means disposed in co-operative relationship with the apertures in the first member and a second plurality of generating means disposed in co-operative relationship with the apertures in the second member, the generating means in the first plurality and the generating means in the second plurality being interrelated and being provided with weighted operating characteristics to produce in the different apertures field densities dependent upon the polarities and magnitudes of the coefficients in a pair of Fourier Series which together represent the arbitrary function to be generated upon progressive displacements between the first and second members and which individually represent arbi-

trary functions to be generated upon progressive relative orientations of the first and second members.

5. In apparatus for generating any desired arbitrary function having only a single value at each position along one of a pair of transverse axes, a first member, a second member movable relative to the first member, a first plurality of generating means disposed at spaced positions along the first member and a second plurality of generating means disposed at spaced positions along the second member, the generating means in the first plurality and the generating means in the second plurality being interrelated and being provided with weighted operating characteristics in accordance with the solution of a plurality of simultaneous equations each involving the weighted operating characteristics of a different harmonic generated by the generating means in the first and second pluralities in relationship to the product of the coefficients of the particular harmonics in a pair of Fourier Series the product of which expresses the arbitrary function to be generated with progressive displacements between the first and second members and which individually express arbitrary functions to be generated with progressive relative orientations along the first member or along the second member.

6. In apparatus for generating any desired arbitrary function having only a single value at each position along one of a pair of transverse axes, a first member made from magnetic material and provided with a plurality of slots at spaced intervals along one periphery, a second member made from magnetic material and disposed for movement relative to the first member and provided with a plurality of slots at spaced intervals along a periphery facing the first member, and a first plurality of windings disposed in the different slots in the first member and a second plurality of windings disposed in the different slots in the second member and provided with conductor densities and wound with polarities in accordance with the solution of a plurality of simultaneous equations each involving the relationship between the conductor densities contributed by the different terms toward a particular harmonic and the coefficient of that harmonic in a Fourier Series which represents the arbitrary function to be generated upon progressive displacements between the first and second members.

7. In apparatus for generating any desired arbitrary function having only a single value at each position along one of a pair of transverse axes, a first member made from magnetic material and provided with a plurality of slots at spaced positions along one of its peripheries, a second member made from magnetic material and disposed for movement along the first member and provided with a plurality of slots at spaced positions along a periphery facing the slots in the first member, the number of slots in the second member being different from the number of slots in the first member for corresponding units of distance, and a first plurality of windings disposed within the different slots in the first member and a second plurality of windings disposed within the different slots in the second member, the windings in the first plurality being disposed and interconnected and the windings in the second plurality being disposed and interconnected to provide in the different slots conductor densities representing the solutions of a pair of Fourier Series the product of which represents the arbitrary function to be generated upon progressive displacements between the first and second members and which individually represent the functions to be generated by the windings in the first plurality and the windings in the second plurality upon such progressive displacements.

8. In apparatus for generating any desired arbitrary function having only a single value at each position along one of a pair of transverse axes, a first member made from magnetic material and provided with first and sec-

ond peripheries and provided with a plurality of slots at spaced intervals along one periphery and provided with a continuous portion at its other periphery for the flow of magnetic flux, a second member made from magnetic material and provided with first and second peripheries with the first periphery facing the slots in the first member and provided with a plurality of slots at spaced intervals along the first periphery and with a continuous portion at its other periphery for the flow of magnetic flux, and a first plurality of windings disposed in the different slots in the first member and a second plurality of windings disposed in the different slots in the second member and provided with conductor densities and wound with polarities in accordance with the convolution product integral of a pair of functions which individually represent the relationship to be generated for progressive orientations along the first and second members and the integrated product of which represents the arbitrary function to be generated for progressive displacements between the first and second members.

9. In apparatus for generating any desired arbitrary function having only a single value at each position along one of a pair of transverse axes, a first member made from magnetic material and provided with a hollow annular configuration and with a plurality of slots at spaced positions along one of its annular peripheries, a second member made from magnetic material and provided with a hollow annular configuration and disposed in concentric relationship to the first member for movement along the first member and provided with a plurality of slots at spaced positions along the annular periphery facing the slots in the first member, the number of slots in the second member being different from the number of slots in the first member, and a first plurality of windings disposed within the different slots in the first member and a second plurality of windings disposed within the different slots in the second member, the windings in the first plurality being disposed and interconnected and the windings in the second plurality being disposed and interconnected to provide in the different slots conductor densities representing the convolution product integral of a pair of functions each of which represents the particular relationship generated for progressive orientations of a different one of the first and second members and the integrated product of which represents the desired arbitrary function to be generated for progressive displacements between the first and second members.

10. In apparatus for generating any desired arbitrary function having only a single value at each position along one of a pair of transverse axes, a first hollow annular member made from magnetically receptive material and provided with slots disposed at spaced positions along the inner periphery of the member and with bars between the slots, a second hollow annular member made from magnetically receptive material and provided with slots disposed at spaced positions along one annular periphery and with bars between the slots, a first plurality of windings disposed in the different slots in the first member and connected in a continuous circuit with the other windings in the plurality, and a second plurality of windings disposed in the different slots in the second member and connected in a continuous circuit with the other windings in the plurality, the first and second pluralities of windings being constructed to provide conductor densities in the different slots in the first and second members for the generation of the desired arbitrary function upon progressive displacements between the first and second members and in accordance with the solution of a plurality of simultaneous equations involving the conductor densities in the different slots and the coefficients of the successive terms in a first Fourier series expressing a particular function to be generated with progressive displacements along the first member and the coefficients of the successive terms in a second Fourier series expressing a particular function to be generated with pro-

gressive displacements along the second member wherein the first and second Fourier series are provided with interrelated characteristics to have their product represent the desired arbitrary function.

11. In apparatus for generating any desired arbitrary function having only a single value at each position along one of a pair of transverse axes, a first member made from magnetically receptive material and provided with an annular configuration having an opening at the center and provided with a plurality of slots spaced annularly around the periphery of the first member, a second member made from magnetically receptive material and provided with an annular configuration having an opening at the center and disposed in concentric relationship to the first member for rotation relative to the first member and provided with a plurality of slots spaced annularly around the periphery of the second member at the periphery facing the first member, and a first plurality of windings disposed in the slots in the first member and a second plurality of windings disposed in the slots in the second member, the windings being connected and being disposed in the different slots in particular patterns to produce in the different slots conductor densities dependent upon the polarities and magnitudes of the coefficients in a Fourier Series representing the arbitrary function to be generated upon progressive displacements between the first and second members, the windings in the first plurality being connected and disposed to produce a resultant value of zero for all of the conductor densities in the different slots in the first member and the windings in the second plurality being connected and disposed to produce a resultant value of zero for all of the conductor densities in the different slots in the second member.

12. In apparatus for generating any desired arbitrary function having only a single value at each position along one of a pair of transverse axes, a first member made from magnetic material and provided with a hollow an-

nular configuration and with a plurality of slots at spaced positions along one of the annular peripheries, a second member made from magnetic material and provided with a hollow annular configuration and disposed in concentric relationship to the first member for rotation relative to the first member and provided with a plurality of slots at spaced positions along the annular periphery facing the slots in the first member, the number of slots in the second member being different from the number of slots in the first member, a first plurality of windings disposed in the different slots in the first member and a second plurality of windings disposed in the different slots in the second member, the windings in the first and second pluralities being constructed to provide the slots with conductor densities dependent upon the magnitudes and polarities of the coefficients in successive terms of a pair of Fourier Series which individually represent progressive relative orientations along the first and second members and which together represent the arbitrary function to be generated for progressive displacements between the first and second members.

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