United States Patent [19]

Weiner

[11] 3,765,773

[45] Oct. 16, 1973

[54] SYSTEMS AND COMPONENTS FOR THE UTILIZATION OF ELECTROMAGNETIC WAVES IN DISCRETE, PHASE-ORDERED MEDIA

[76] Inventor: Melvin Milton Weiner, 56 Marcellus Dr., Newton Centre, Mass. 43614

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[21] Appl. No.: 78,056

[52] U.S. Cl. 356/114, 356/111, 350/96
[51] Int. Cl. G01n 21/40, G01h 9/02
[58] Field of Search 350/96; 356/114, 356/106

[56] References Cited

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Primary Examiner—Ronald L. Wibert
Assistant Examiner—Conrad Clark
Attorney—Charles Hieken

[57] ABSTRACT

Systems and components are disclosed for the utilization of electromagnetic waves whose phase difference, between lattice planes in discrete phase-ordered media, is appreciable in the forward-scattered direction.

8 Claims, 30 Drawing Figures

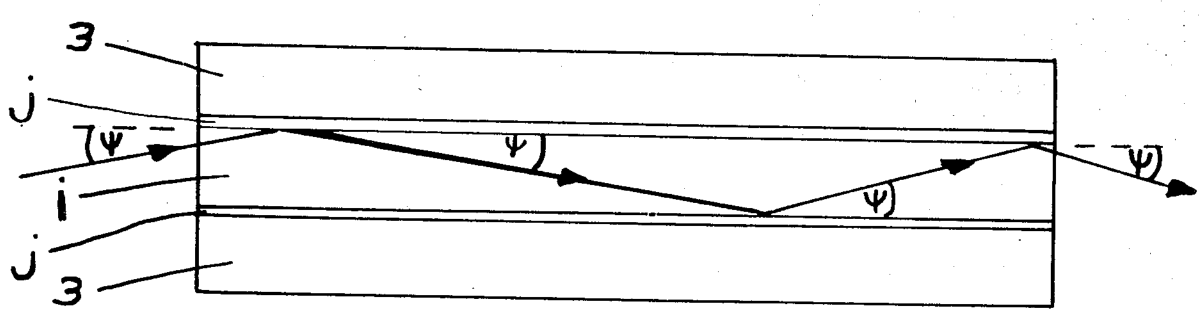


FIG. 1a

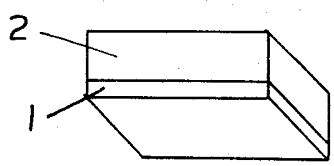


FIG. 1b

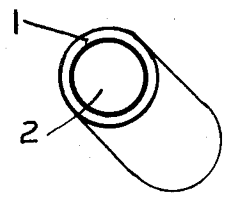


FIG. 1C

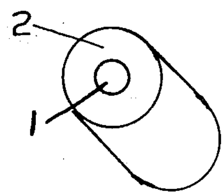


FIG. 1d

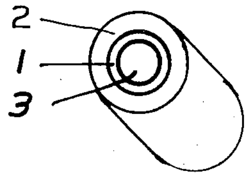


FIG. 1e

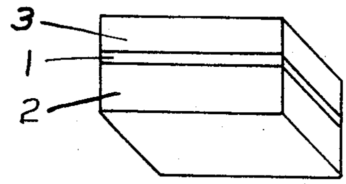


FIG. 2a

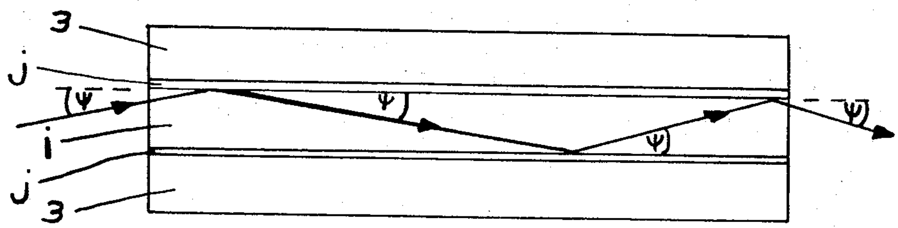


FIG. 2b

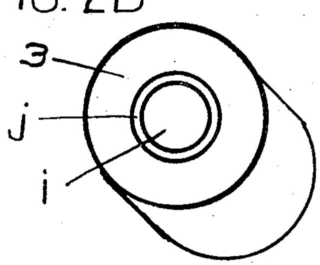


FIG. 2C

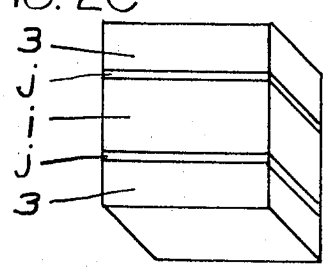


FIG. 3a

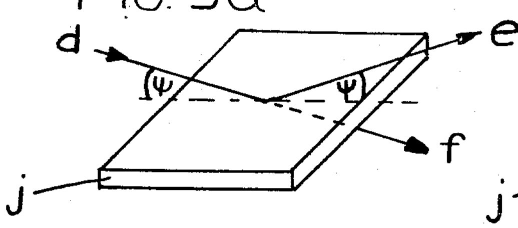


FIG. 3b

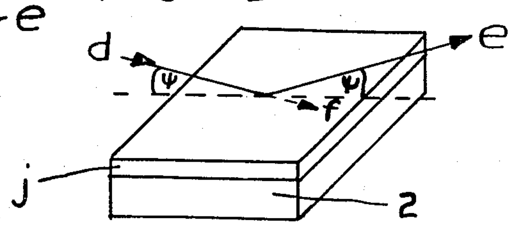


FIG. 4

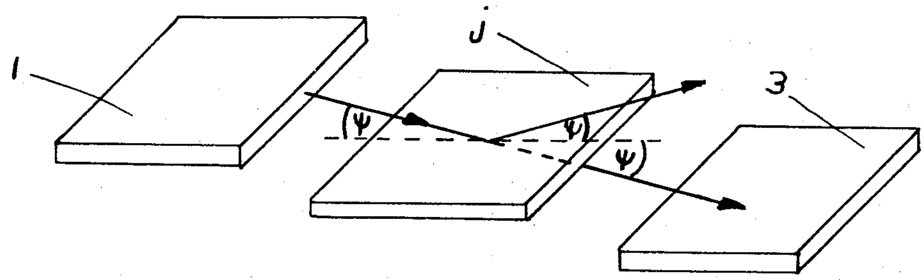


FIG. 5

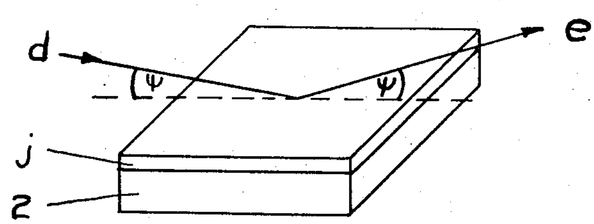


FIG. 6

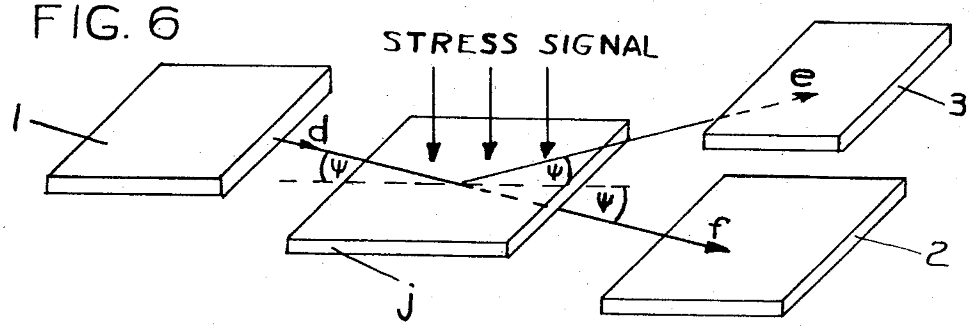


FIG. 7

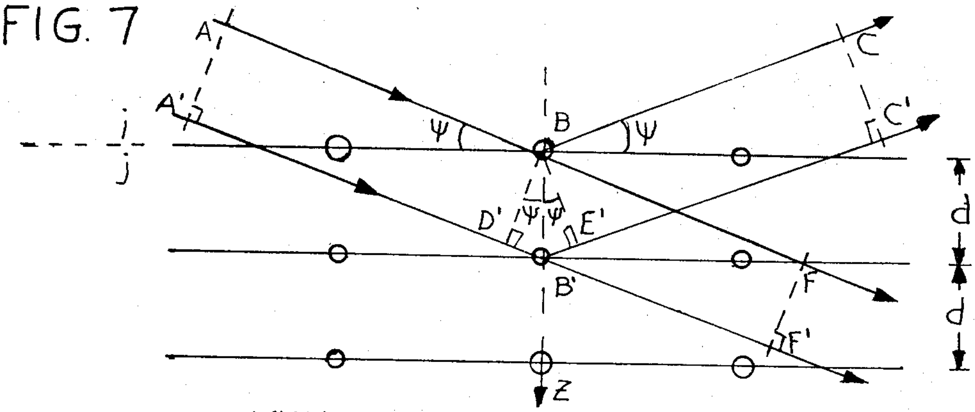


FIG. 8a

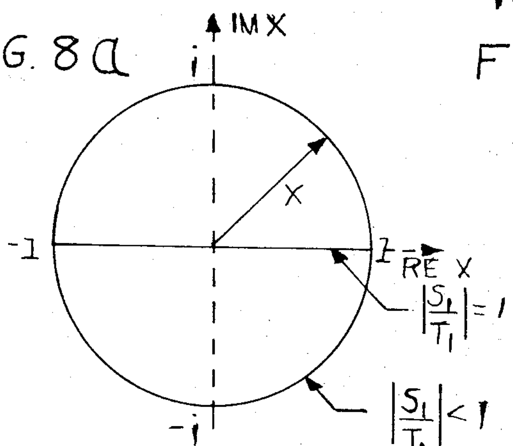


FIG. 8b

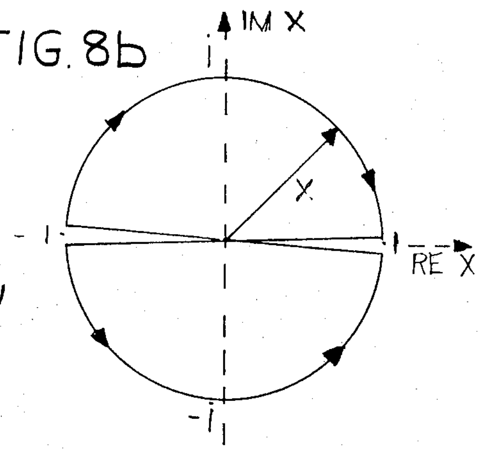
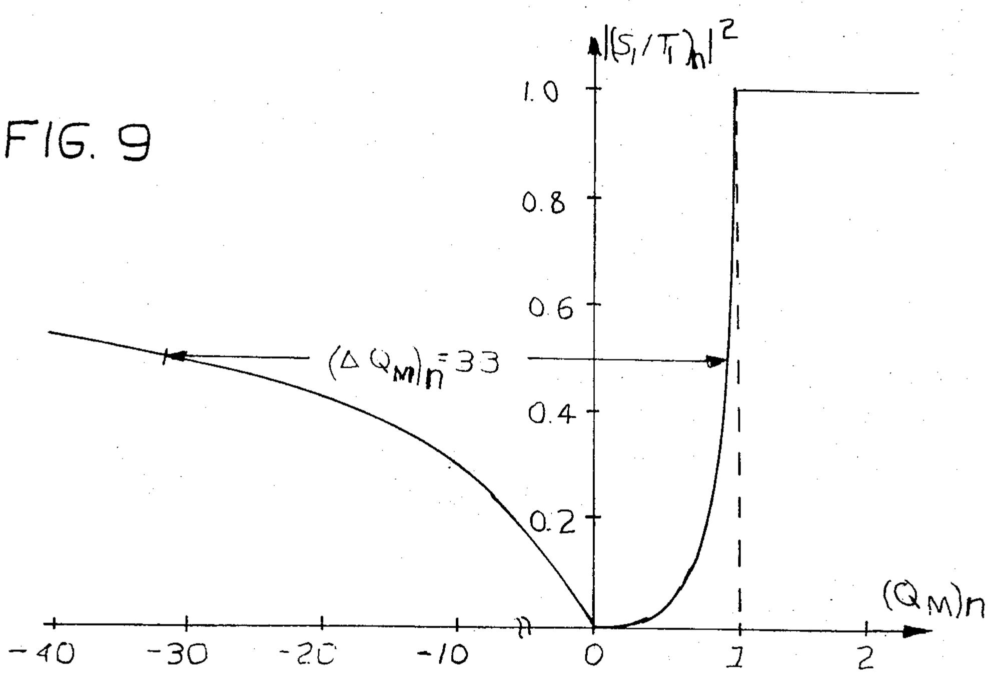


FIG. 9



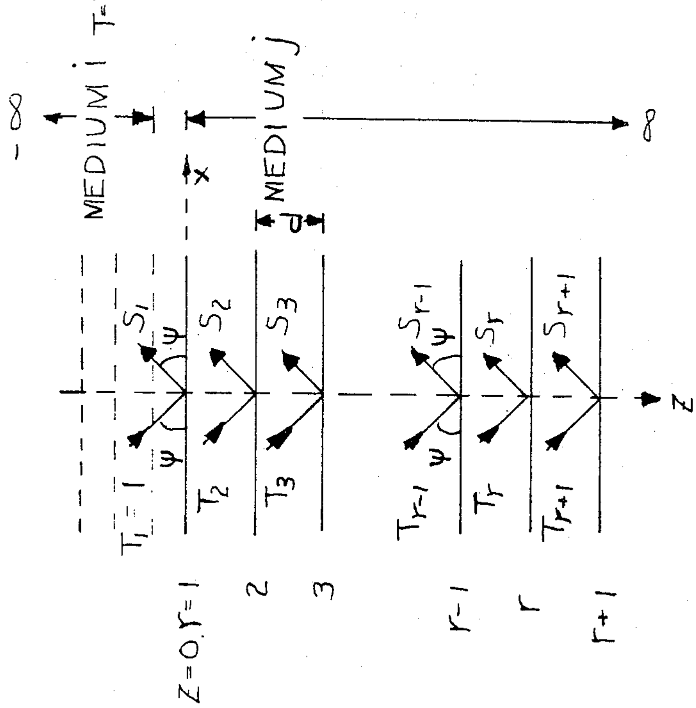


FIG. 10a

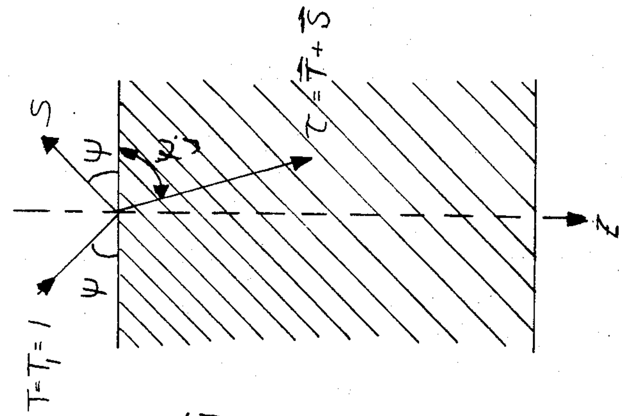


FIG. 10b

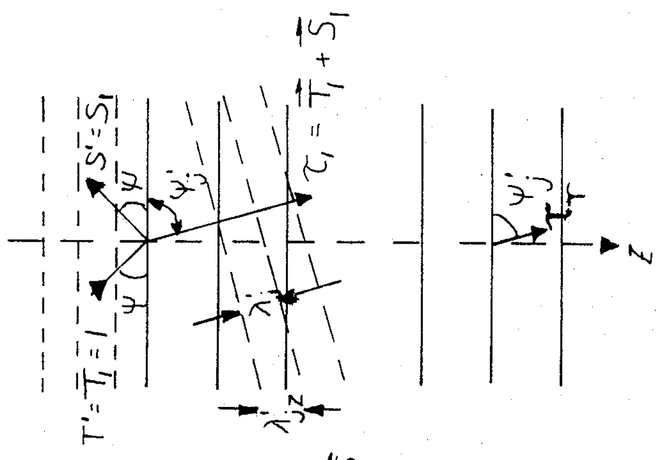


FIG. 10c

FIG. 11

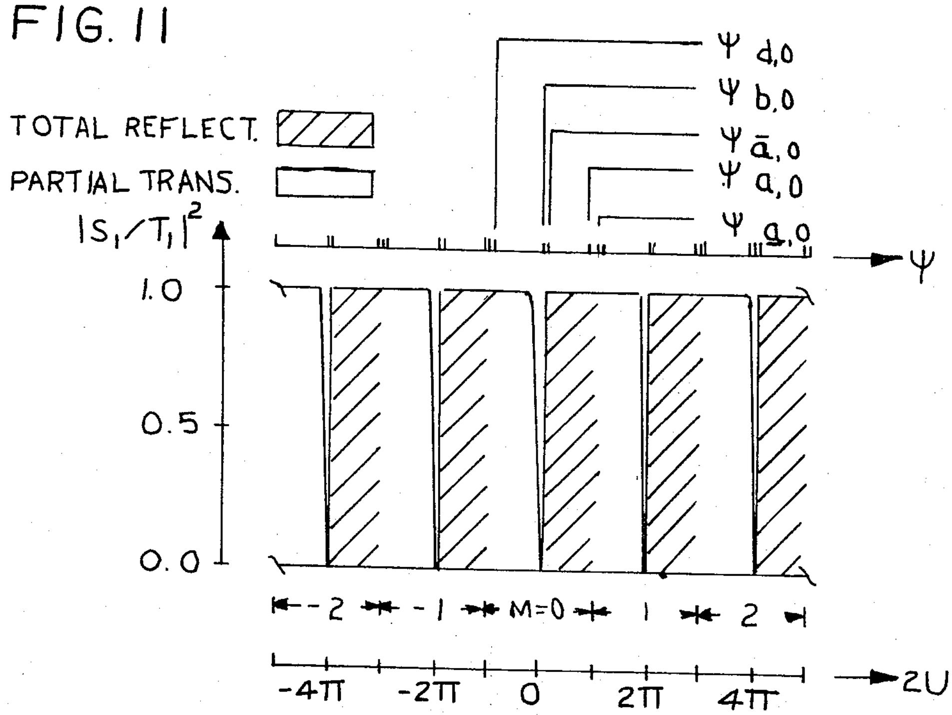
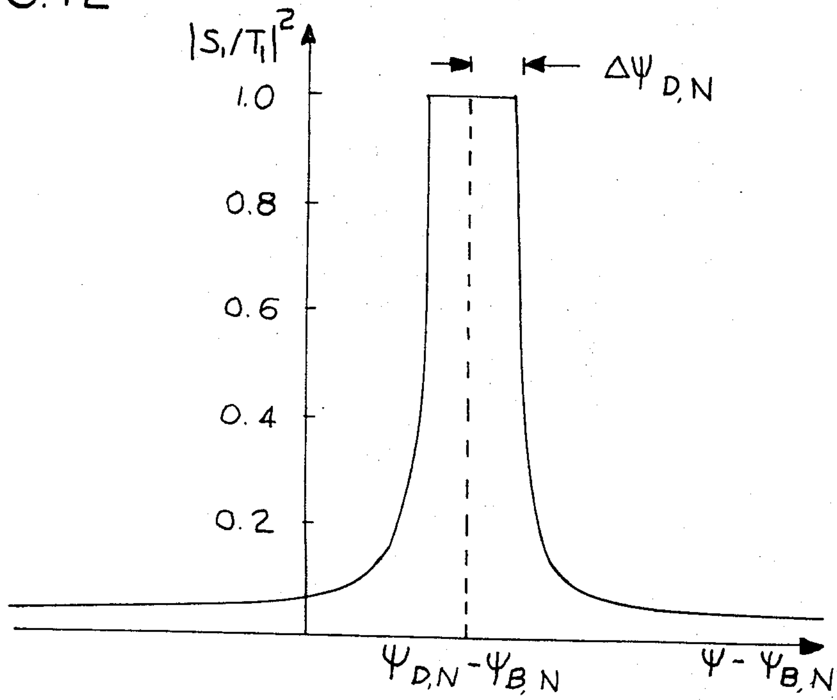

TOTAL REFLECT.
PARTIAL TRANS.
$|S_1/T_1|^2$
1.0
0.5
0.0
Ψ d,0
Ψ b,0
Ψ ā,0
Ψ a,0
Ψ a,0
Ψ
-2
-1
M=0
1
2
-4π
-2π
0
2π
4π
2U

FIG. 12


$|S_1/T_1|^2$
1.0
0.8
0.6
0.4
0.2
$\Delta\Psi_{D,N}$
$\Psi_{D,N}-\Psi_{B,N}$
$\Psi-\Psi_{B,N}$

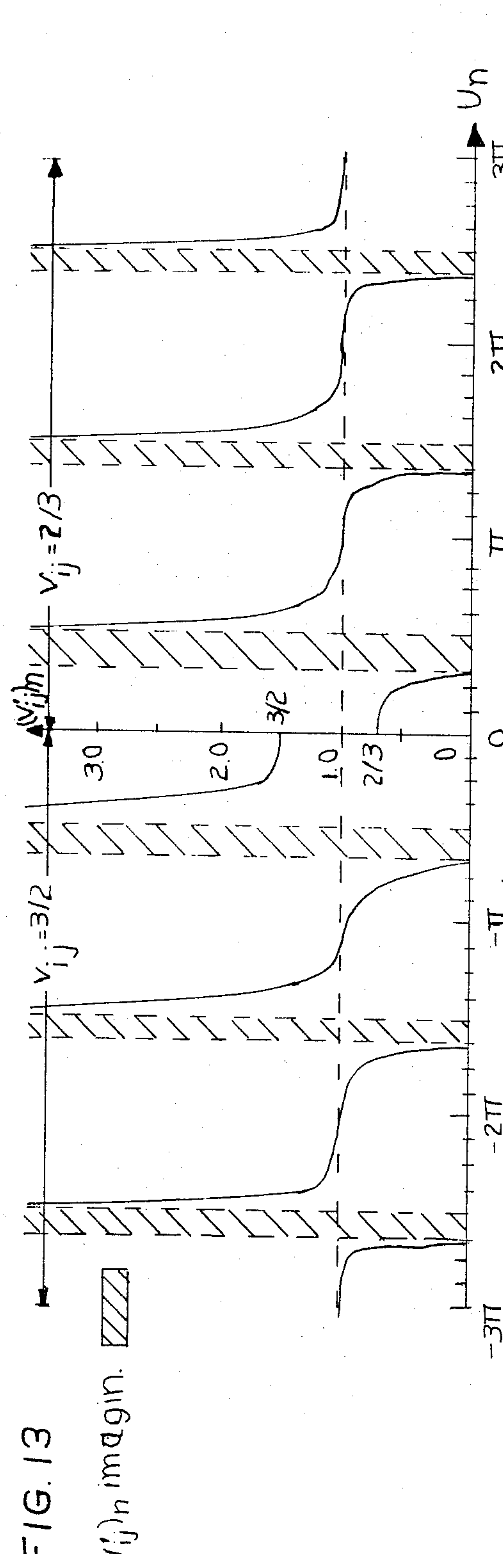

Un
3π
2π
π
0
-π
-2π
-3π
30
2.0
1.0
0
3/2
2/3
M = 1
M = 0
M = -1
(ν'ij)n imagin.

FIG. 13

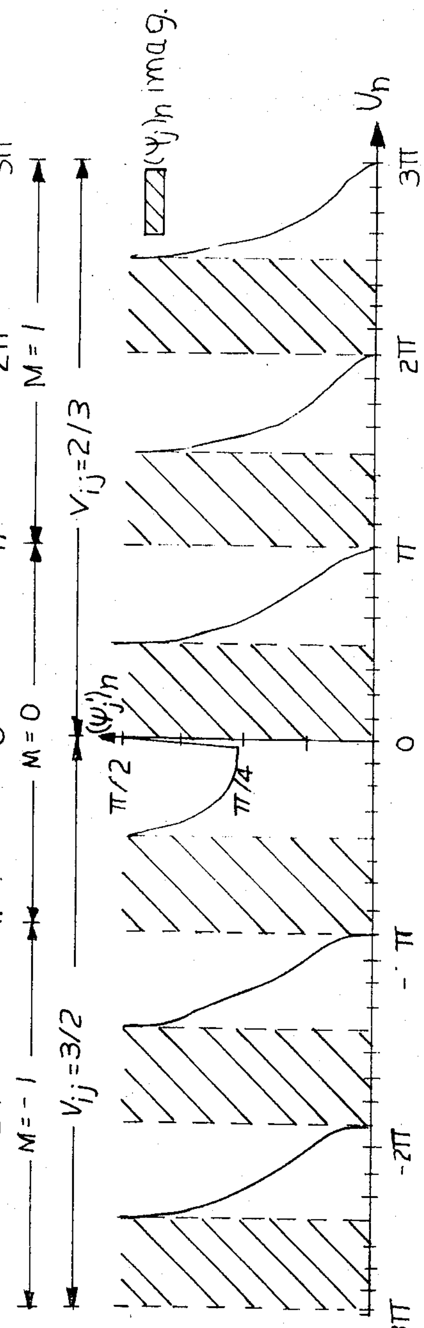

Un
3π
2π
π
0
-π
-2π
-3π
π/2
π/4
(ψ'j)n imag.

FIG. 14

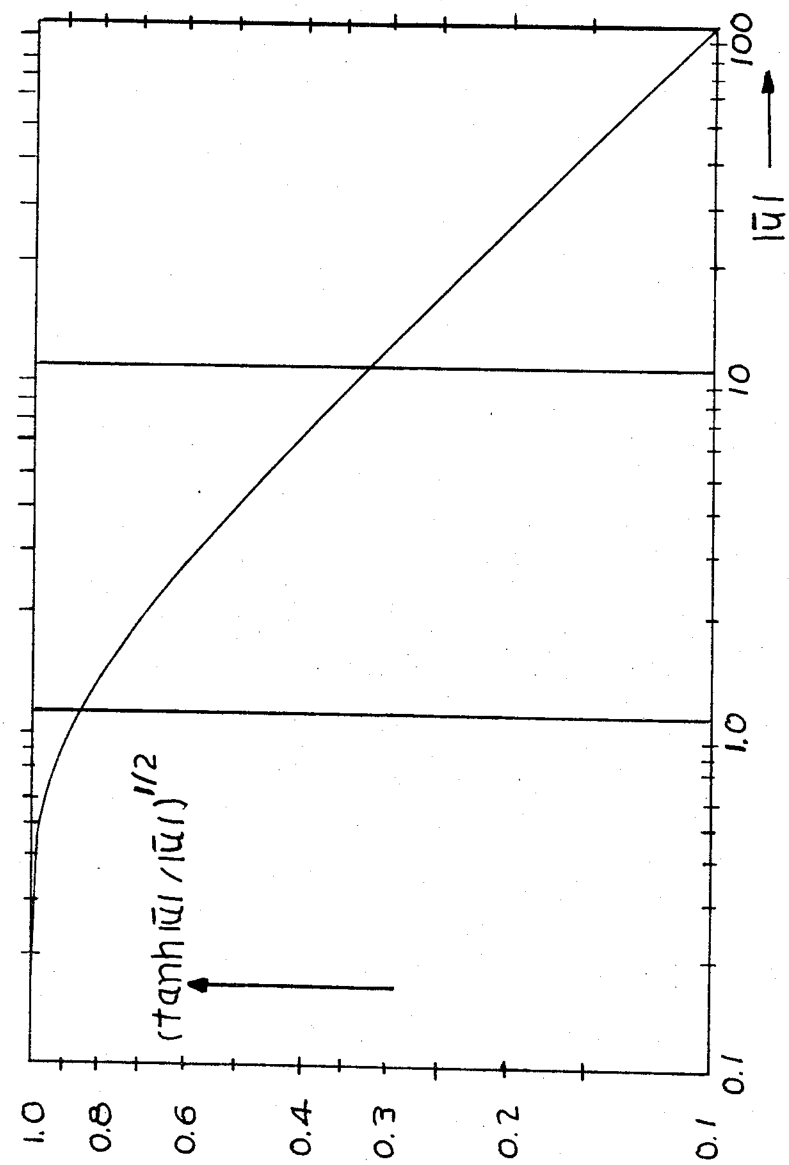


FIG. 15

FIG. 16

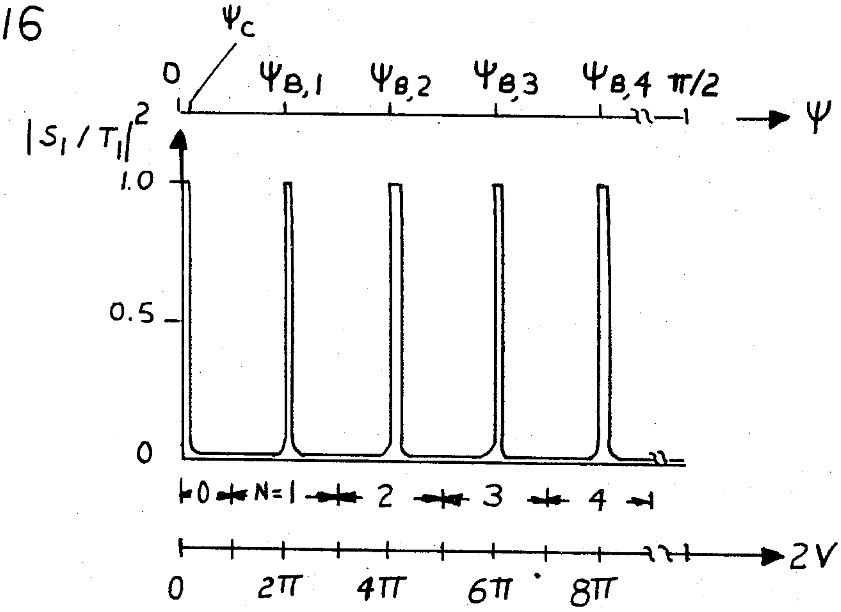
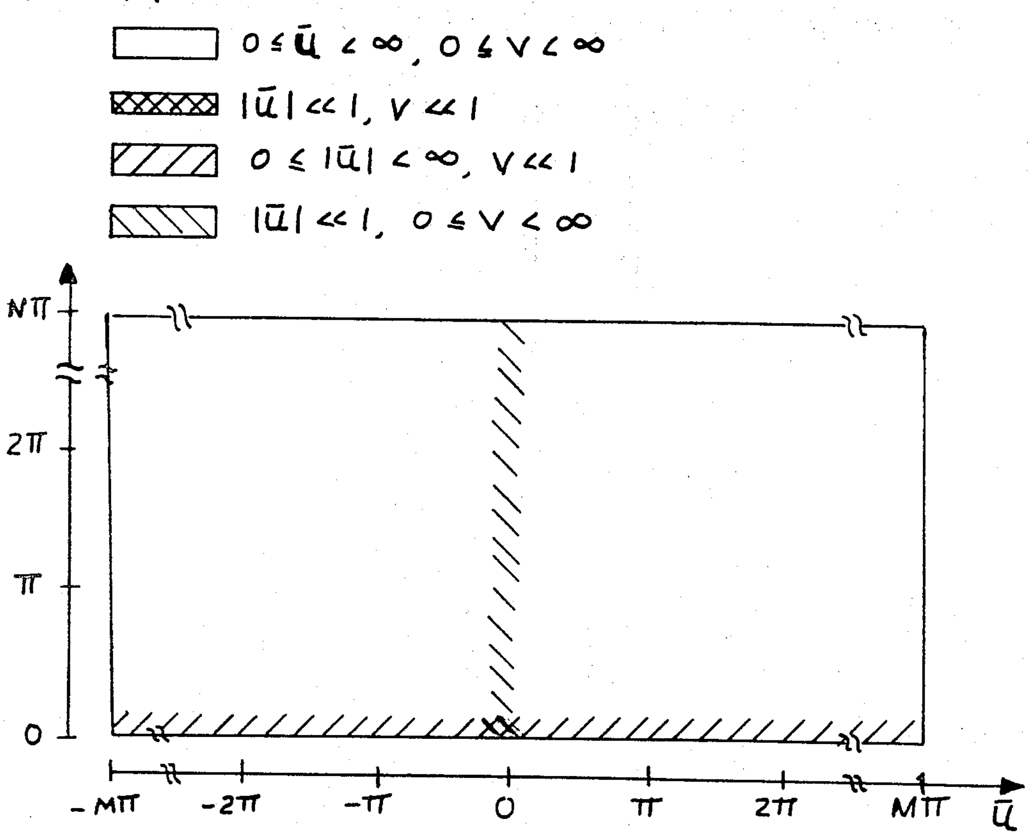

ψ_c
0
$\psi_{B,1}$
$\psi_{B,2}$
$\psi_{B,3}$
$\psi_{B,4}$
$\pi/2$
ψ
$|S_1/T_1|^2$
1.0
0.5
0
0
N=1
2
3
4
2V
0
2π
4π
6π
8π

FIG. 17


$0 \le \bar{u} < \infty$, $0 \le V < \infty$
$|\bar{u}| \ll 1$, $V \ll 1$
$0 \le |\bar{u}| < \infty$, $V \ll 1$
$|\bar{u}| \ll 1$, $0 \le V < \infty$
$N\pi$
2π
π
0
$-M\pi$
-2π
$-\pi$
0
π
2π
$M\pi$
$\bar{u}$

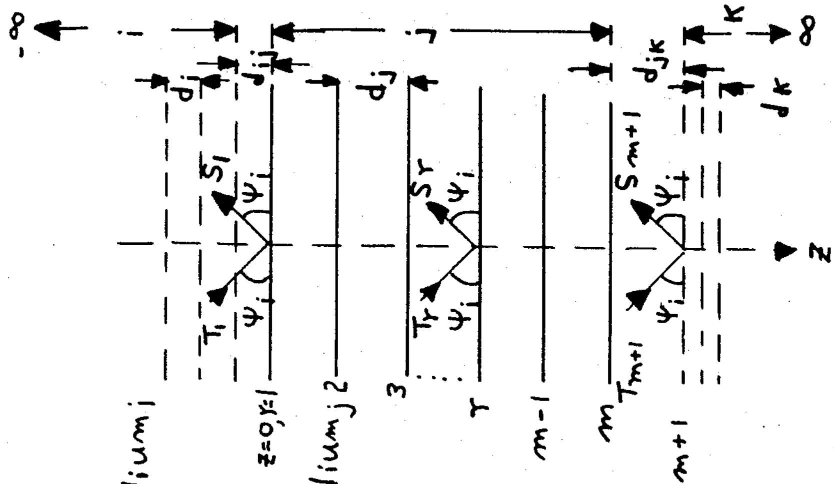


FIG. 18 C

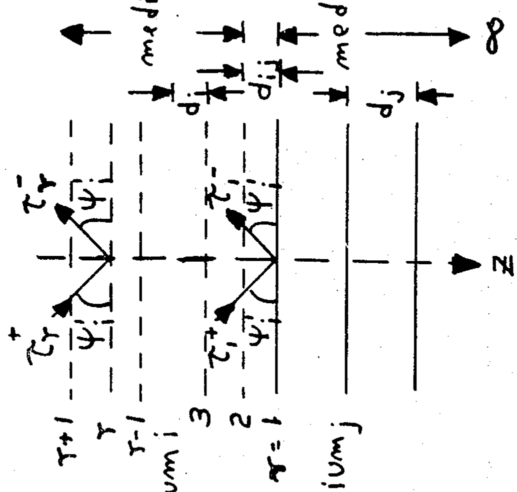


FIG. 18 b

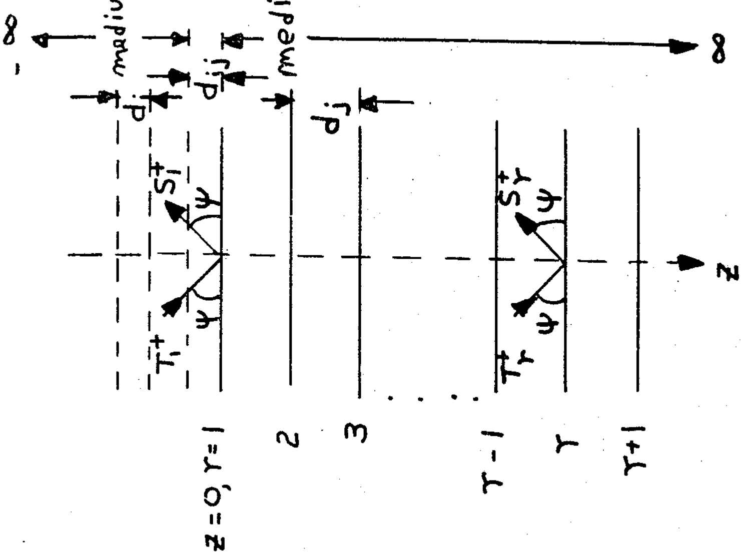


FIG. 18 a

SYSTEMS AND COMPONENTS FOR THE UTILIZATION OF ELECTROMAGNETIC WAVES IN DISCRETE, PHASE-ORDERED MEDIA

BACKGROUND OF THE INVENTION

This invention relates to the utilization of the transmission, reflection, dispersion, and polarization properties of electromagnetic waves, over the entire electromagnetic spectrum, in discrete, phase-ordered media. The theory of electromagnetic waves in discrete media is disclosed by the applicant in Appendix A. The concept and definition of "phase-ordered" media are also disclosed in Appendix A. Phase-ordered media include continuous media, single crystals, liquid crystals, polycrystalline and amorphous media in which the phase differences between scattering centers are small, and artificial lattices consisting of media in which obstacles are periodically imbedded or media which are periodically modulated by an elastic wave or other incident wave. Specific properties of interest in this invention fall into one of two categories. The first category includes the transmission bands, reflection bands, frequency dispersion, and birefringence which occur at sufficiently small grazing angles of incidence in discrete, phase-ordered dielectric media. The second category includes the increased skin depth and reduced conduction loss of electromagnetic waves at arbitrary angles of incidence in discrete, phase-ordered metallic media.

In particular, this invention relates to metallic radiation guides, such as wires, transmission lines, waveguides, and cavities, in which travelling or standing electromagnetic waves propagate in well-defined modes with minimum radiation loss. This invention also relates to dielectric radiation guides, such as fibers and thin films, in which electromagnetic waves propagate by successive multiple reflection with minimum radiation loss. This invention also relates to systems and components for control, communication, display, data-processing, information storage, and detection in which precise filtering, switching, or modulation of the direction, wavelength, amplitude, or polarization of the incident electromagnetic radiation is desired. This invention further relates to systems for the investigation of lattice structure and lattice defects of materials. Still further, this invention relates to the detection of externally applied energy which modifies the lattice structure or the electrical properties of a discrete, phase-ordered medium.

The present state-of-the-art of most electromagnetic systems and components is characterized by wave interactions in which the phase difference between scattering centers of the interacting medium can be neglected or assumed to be zero. The medium is therefore usually referred to as being "continuous." The wave interactions for continuous media can differ significantly from the wave interactions for discrete phase-ordered media in which the phase difference between scattering centers is appreciable. A detailed comparison, of the properties of electromagnetic waves in continuous and discrete phase-ordered media, is given in Appendix A. This invention relates to a specific class of wave interactions in discrete phase-ordered media. This invention specifically relates to waves whose phase difference, between lattice planes in discrete phase-ordered media, is appreciable in the forward direction. (see Appendix A, Conditions $V << 1, 0 \leq |\bar{u}| < \infty$). This invention should not be confused with wave interactions in the present state-of-the-art of periodic media. One such class of wave-interactions is characterized by Bragg reflection in which the phase difference between lattice planes is appreciable in the reflected direction (see Appendix A,). Another class of wave interactions in the present state-of-the-art of periodic media is characterized by wavelengths which are large compared to the spacings between scattering centers of an artificial lattice (see W. E. Kock, "Metallic Delay Lenses" Bell Syst. Techn. J. 27, 58, 1948). A further class of wave interactions in the present state-of-the-art of periodic media is restricted to waves which are incident at normal incidence on artificial dielectrics (see S. B. Cohn, J. App. Phys. 20, 257, 1949; 21, 674, 1950; 22, 628, 1951; see also H. S. Bennett, "The Electromagnetic Transmission Characteristics of the Two-Dimensional Lattice Medium," J. App. Phys. 24, 785, 1953), In this latter case of wave interactions, the phase difference between lattice planes can be appreciable in the forward direction. Also related to this latter class of wave interactions is the propagation of waves in periodic-loaded transmission lines in which the incident and exit directions of propagation are known (as in the case of normal incidence). (See L. Brillouin, Wave Propagation in Periodic Structures, New York: Dover, 1953, Chapters 9, 10, and Appendix).

This invention relates to two classes of wave interactions in which the phase differences between lattice planes is appreciable in the forward direction and which have not been utilized in the present state-of-the-art of electromagnetic systems and components. The first class of wave interactions related to this invention consists of electromagnetic waves incident at small grazing angles of incidence on a discrete phase-ordered dielectric. The second class of wave interactions related to this invention consists of electromagnetic waves incident at an arbitrary angle of incidence on a discrete phase-ordered metal.

The unique electromagnetic properties related to this invention for the first class of wave interactions,in phase-ordered dielectrics,are:

a. The effective phase velocity of wave propagation is a function of the angle of incidence, polarization, and incident wavelength for a given dielectric constant and lattice spacing of the medium.
b. Narrow transmission bands occur for angles of incidence sufficiently small.
c. Total reflection occurs in relatively wide bands, for angles of incidence sufficiently small, regardless of whether the relative phase velocity is less than or greater than unity.

The unique electromagnetic properties related to this invention for the second class of wave interactions, in phase-ordered metals, are:

a. The effective phase velocity of wave propagation is weakly dependent upon the DC conductivity and is strongly dependent upon the lattice spacing for media of large DC conductivity.
b. The effective skin depth is larger (and the effective surface resistivity is smaller) than for continuous metals.

One component of this invention, common to all claims and embodiments, is a phase-ordered medium, whether it be a dielectric or a metal. A phase-ordered medium is one which satisfys the conditions of Equation 2, 7 of Appendix A and includes those media which have been identified above. If the incident electromagnetic radiation penetrates to only a limited effective distance within the medium, then it is understood that the conditions of Equation 2, 7 need only be satisfied by that portion of the medium to which the incident radiation penetrates. Therefore a medium with only short range order may be phase-ordered. This is particularly applicable for radiation incident at small grazing angles of incidence because an appreciable percentage of the incident radiation may be reflected by a single lattice plane. A second example of a phase-ordered medium which may have only short range order is radiation incident on metals with a skin depth of penetration comparable to a few lattice spacings. The scattering centers of a phase-ordered medium j may consist of either

a. atomic or sub-atomic particles, such as electrons, atoms, ions, molecules, unit cells of a crystal, which satisfy the conditions of Equation **2, 7** of Appendix A;

b. obstacles which are periodically imbedded in a dielectric to form an artificial lattice;

c. atomic or sub-atomic particles of a medium which is periodically and spatially modulated by an elastic wave or other wave propagating in the medium, such as an acoustical wave which spatially modulates a liquid crystal.

A phase-ordered medium *j* may exist as a solid, liquid, gas, or plasma provided that the conditions of Equation 2.7 of Appendix A are satisfied. Although the applicant defines and introduces in Appendix A the concept of phase-ordered media, this application does not claim the invention of phase-ordered media nor does it claim the invention of new forms of phase-ordered media. Rather, this application claims new and novel ways of utilizing the properties, disclosed by the applicant in Appendix A, of electromagnetic waves in phase-ordered media.

In defining a phase-ordered dielectric and a phase-ordered metal, it is convenient to refer to the parameter $v_{ij}=v_i/v_j=(\mu_j \epsilon_j/\mu_i\epsilon_i)^{1/2}$ = relative wave phase velocity for continuous media where

v_i = wave phase velocity in medium *i* if medium *i* were continuous

v_j = wave phase velocity in medium *j* if medium *j* were continuous

μ_j, μ_i = permeabilities of the respective media if the respective media were continuous

ϵ_j, ϵ_i = permittivities of the respective media if the respective media were continuous.

The parameter v_{ij} is generally a complex number since most media have some absorption loss. If Re v_{ij} >> Im v_{ij} (the real part of v_{ij} is much larger than the imaginary part of v_{ij}) then medium *j* shall be referred to as a "dielectric" with respect to medium *i*. If Im v_{ij} >> Re v_{ij}, then medium *j* shall be referred to as a metal with respect to medium *i*. A phase-ordered dielectric is therefore, for purposes of this invention, a medium *j* which satisfys the conditions of Equation 2.7 of Appendix A and for which (Im v_{ij}/Re v_{ij}) << 1. A phase-ordered metal, for purposes of this invention, is a medium j which satisfys the conditions of Equation 2.7 of Appendix A and for which (Im v_{ij}/Re v_{ij}) >> 1. "Total" reflection or "total" transmission by a dielectric shall be interpreted as referring to an idealized dielectric with Im v_{ij}=0.

BRIEF SUMMARY OF THE INVENTION

According to the invention, there is apparatus for selectively propagating waves in a direction having a component parallel to an interface comprising means defining discrete phase-ordered media along the interface characterized by a wave refraction constant, such as the dielectric constant, permittivity, permeability or index of refraction, and a spacing d between lattice planes generally parallel to the interface, a source of waves, and means for establishing the wavelength, polarization and angle of incidence upon the inter-face of the waves to selectively propagate the waves in a direction having a component parallel to the interface in a predetermined manner controlled by the wave refraction constant and the spacing d and establish a substantial phase difference between wave fronts at adjacent ones of the lattice planes in the forward scattered direction at least of the order of a radian.

One embodiment of the invention is a metallic radiation guide, for the propagation or containment of electromagnetic waves in well-defined modes, whose conductors and boundary walls comprise a discrete phase-ordered metal. The geometry and configuration of the radiation guide may comprise that of any existing wire, transmisssion line, waveguide, cavity, or electrical circuit which guides electromagnetic radiation by means of one or more metallic conductors. The claim of this invention is that the metallic conductors (media of large DC conductivity) of the radiation guide are discrete and phase-ordered. A preferred embodiment of this invention is that the phase-ordered metals have maximum DC conductivity with maximum spacing of the lattice planes parallel to the guide wall. Such a preferred embodiment has minimum surface resistivity in metallic radiation guides of well-defined modes of propagation (see Equation 4.67 and Table 7 of Appendix A). Metallic conductors, of present radiation guides of well-defined modes of propagation, are usually electroformed or electroplated metals consisting of randomly oriented crystals with an effective lattice structure of large phase disorder. Such metallic conductors are usually treated in practice as if they were continuous metals with zero phase difference between scattering centers. Table 7 of Appendix A demonstrates that the surface resistivity for radiation guides of this invention, which utilize discrete, phase-ordered metals, is less than for those which utilize continuous metals. This invention is applicable to metallic radiation guides in which the radiation is guided in well-defined modes and is not applicable to metallic radiation pipes in which the radiation propagates by multiple reflection in undefined modes. The cross-section of the radiation guide and of the phase-ordered metal may be of any geometry, including rectangular and circular geometries, appropriate for a given mode of propagation. The phase-ordered metal should have a thickness at least equal to the effective skin depth given by Equation 4.66 of Appendix A. The discrete, phase-ordered metal may adjoin or be clad by dielectric media which are internal or external to the radiation guide and which are not necessarily phase-ordered. The discrete, phase-ordered metal may also adjoin or be mechanically supported by a metallic medium which is external to the radiation guide and which is not necessarily phase-ordered. The metallic radiation guide of this invention is particularly well-suited to microminiaturized, high frequency electric circuits such as microcircuits and integrated semiconductor circuitry because:

a. only small amounts of phase-ordered metal are required;

b. minimum surface resistivity is required because of the small size of the conductors; and

c. the phase-ordered metal is compatable and may be integrated with the single-crystalline media associated with microminiaturized circuitry.

A second embodiment of the invention is a dielectric radiation pipe, for the propagation of electromagnetic waves by successive multiple reflection within certain defined bands of grazing angles of incidence, whose boundary walls comprise a discrete, phase-ordered dielectric. If ψ is the grazing angle of incidence, then the defined bands of grazing angles of incidence are $\psi_{\underline{a},M} \leq \psi \leq \psi_{\underline{a},M}$ where $\psi_{\bar{a},M}$ and $\psi_{\underline{a},M}$ are given by Equation 4.43*c* of Appendix A and are the angles corresponding to the boundaries of the total reflection regions for electromagnetic waves incident on discrete, phase-ordered dielectrics. The most common form of the present state-of-the-art of dielectric radiation pipes comprises a dielectric fiber or clad dielectric fiber through which radiation propagates by multiple reflection at grazing angles of incidence less than the critical angle ψ_c of total reflection. In the present state-of-the-art, the cladding medium j is assumed to have the properties of a continuous dielectric whose index of refraction n_j is less than the index of refraction n_i of the fiber i. The claims of this invention are that the cladding medium j is a discrete, phase-ordered dielectric and that the grazing angles of incidence are restricted to the total reflection bands of the discrete, phase-ordered dielectric. The index of refraction, of the cladding medium j of this invention, may be greater than the medium i about which it is clad. A preferred embodiment of this invention is that the medium i, through which the radiation propagates, is free space and that the cladding medium j is a discrete, phase-ordered dielectric whose walls serve as the boundary of the free-space medium i. The cross-section of the radiation guide may be of any geometry, including tubular and slab geometries, in which the radiation can propagate by successive multiple reflection at constant angles of incldence. The discrete, phase-ordered medium j should be of sufficient thickness, as determined by Section 5.0 of Appendix A, so that total or almost total reflection is achieved. The discrete, phase-ordered medium j may be mechanically supported by and adjoin a dielectric or metallic medium which is external to the walls of the radiation guide. The dielectric radiation guide of this invention is particularly well-suited to present applications which utilize thin films and fibers as radiation guides. The conditions for total reflection at the surface of a discrete, phase-ordered dielectric may be satisfied for sufficiently small grazing angles of incidence, although it should be clear from Equation 4.43*c* of Appendix A that not all small grazing angles of incidence satisfy the conditions for total reflection. The advantages of this invention over the present state-of-the-art are

a. that the cladding medium j may be of larger index of refraction than the fiber medium i;

b. the radiation is reflected more completely at the surface of the cladding medium j for small grazing angles of incidence;

c. the absorption loss of the fiber medium i can be eliminated completely since the medium i can be free space.

It should be understood that the index of refraction, of either the fiber medium i or the cladding medium j, may be varied by electrical or mechanical means for the purpose of filtering, switching, or modulating the incident radiation. It should be further understood that the conditions for total reflection at the surface of a discrete phase-ordered dielectric are a function of the direction, wavelength, and polarization of the incident radiation. Therefore, the dielectric radiation guide of this invention can be used to filter the direction, wavelength, or polarization of the incident radiation. It should be still further understood that several dielectric radiation guides of this invention may be arranged in an array or bundle for purposes of control, communication, display, data-processing, information storage, or detection of the incident radiation. For such a case, a preferred embodiment of this invention is a swiss cheese configuration in which the "cheese" is the discrete, phase-ordered dielectric j common to all of the radiation guides and the "holes in the cheese" are the media i through which the radiation propagates.

A third embodiment of the invention is an electromagnetic band pass filter, comprising a phase-ordered dielectric material, which transmits very narrow bands of radiation of a given wavelength and polarization at specific small grazing angles of incidence. The angles of incidence ψ (and the corresponding wavelengths and polarization), at which the filter is totally transmitting, are $\psi = \psi_{b,M}$, $M = \pm 1, \pm 2, \ldots$, where $\psi_{b,M}$ is given by Equation 4.42*b* of Appendix A. The 3 db transmission bandwidths $\Delta\psi_{b,M}$, associated with the angles $\psi_{b,M}$ of total transmission, are given, for normal polarization, by Equation 4.46 of Appendix A. Since the angle $\psi_{b,M}$ is a function of the polarization, the band pass filter may also serve as a polarizer. In the present state-of-the-art, total transmission in a dielectric medium is achieved either for the relative index of refraction $\nu_{ij} = 1$ or for parallel polarization at Brewster's angle $\psi_{Brewster} = (\psi_{b,O})_p = \cot^{-1}\nu_{ij}$ (see Equation 4.42*a* of Appendix A). Brewster's angle $(\psi_{b,O})_p$ is usually a relatively large grazing angle of incidence. The claim of this invention is that the dielectric medium is phase-ordered at small grazing angles of incidence and in particular at the angles $\psi_{b,M}$ so that the dielectric behaves as a narrow band pass filter. A preferred embodiment of this invention is a discrete, phase-ordered dielectric upon which radiation is incident from free space and from which the radiation, within the pass band of the filter, is transmitted into free space where it may be detected or further processed according to the particular application. Another preferred embodiment of this invention is a discrete phase-ordered dielectric which is adjoined by an absorbing detector at the exit surface of the dielectric. For both of these embodiments, the thickness of the phase-ordered dielectric does not appreciably affect the transmission properties of the dielectric. The phase-ordered dielectric may be of arbitrary geometry. However, a rectangular slab or disc geometry is preferred. This invention is particularly suited to applications requiring thin films and miniaturized circuitry. It should be understood that the filter may be one component of a larger system of components for purposes of control, communication, display, data-processing, information storage, or detection of the incident radiation. It should be further understood that the index of refraction or the effective lattice spacing of the phase-ordered dielectric may be designed to be variable with the application of an externally applied electrical voltage or mechanical stress. If such is tha case, then the filter acts as a switch or modulator of the transmitted radiation.

A further embodiment of the invention is an electromagnetic system, for the investigation of the lattice structure and lattice defects of materials, comprising a source of electromagnetic radiation, a discrete, phase-ordered dielectric whose lattice structure is to be investigated and upon which the radiation is incident at small grazing angles of incidence, and a detector of the narrow band or bands of radiation which are transmitted by the phase-ordered dielectric. In this embodiment of the invention, the phase-ordered dielectric, whose lattice structure is being investigated, behaves as the filter of the preceding embodiment. Therefore the transmission bandwidths are $\Delta\psi_{b,M}$ at incident grazing angles $\psi_{b,M}$ of total transmission where $M = \pm1, \pm2, \ldots$. The effective lattice spacing, of the lattice planes parallel to the interface, is determined from Equation 4.42*b* of Appendix A, assuming that the incident wavelength, angle of incidence, and the relative index of refraction are determined independently. The reduced transmission, which is detected at an expected angle $\psi_{b,M}$ of total transmission, is an indication of the lattice defects or phase disorder of the dielectric. In the present state of the art, the investigation of lattice structure is usually achieved by either x-ray Bragg reflection, electron diffraction, or neutron diffraction. The latter two methods utilize particle bombardment of the specimen whereas the first method utilizes electromagnetic radiation of the specimen at sufficiently large angles of incidence $\psi_{B,N}$ (see Equation 4.74*c* of Appendix A) so that the phase difference between lattice planes is approximately 2π radians in the reflected direction. The claim of this invention is that electromagnetic radiation, which is not necessarily restricted to x-ray wavelengths, is incident on the specimen at sufficiently small grazing angles of incidence $\psi_{b,M}$ so that the phase difference between lattice planes is approximately 2π radians in the forward scattered direction. A preferred embodiment of the invention comprises a collimated, monochromatic source of optical radiation, a slab of the phase-ordered dielectric specimen, and an optical detector of the radiation transmitted by the specimen. The wavelength or the angle of incidence may be varied through a calibrated range of values. Embellishments such as Michelson interferometry may also be employed.

A fifth embodiment of the invention is an electromagnetic mirror, comprising a discrete phase-ordered dielectric material, which reflects radiation for specific bands of small grazing angles of incidence. The specific bands of grazing angles of incidence are $\psi_{a,M} \leq \psi \leq \psi_{\bar{a},M}$ which are defined in the description of the second embodiment of this invention. The electromagnetic mirror of this fifth embodiment behaves in the same manner as the boundary walls of the dielectric radiation pipe of the second embodiment of the invention. In the present state-of-the-art of dielectric mirrors, the dielectric material is usually either a multilayer of dielectrics whose thicknesses are critical with respect to the incident wavelength or is a dielectric which is assumed to be continuous with suitably small index of refraction so that total reflection can occur at grazing angles of incidence less than the critical angle of total reflection. The claims of this invention are that the dielectric is phase-ordered and that the grazing angles of incidence are restricted to the total reflection bands $\psi_{a,M} \leq \psi \leq \psi_{\bar{a},M}$ of the discrete phase-ordered dielectric. The index of refraction of the phase-ordered dielectric may be greater or smaller than that of the medium from which the radiation is incident. A preferred embodiment of this invention is that the radiation is incident from free space on a phase-ordered dielectric which may adjoin, for purposes of mechanical support, a dielectric or metallic medium. The surface of the mirror may be planar or curved depending upon whether the focal point of the mirror is infinite or finite. The phase-ordered dielectric should be of sufficient thickness, as determined by Section 5.0 of Appendix A, so that total or almost total reflection is achieved. The relative index of the refraction of the phase-ordered dielectric may be varied by electrical or mechanical means for the purpose of filtering, switching, or modulating the incident radiation. Since the reflection bands of the mirror is a function of the direction, wavelength, and polarization of the incident radiation, the mirror of this invention may be used as a selective filter of the direction, wavelength, or polarization of the incident radiation.

A sixth embodiment of the invention is an electromagnetic transducer of externally applied stress, comprising a source of electromagnetic radiation, a discrete, phase-ordered dielectric upon which the radiation is incident at the narrow pass bands of the dielectric at sufficiently small grazing angles of incidence, detectors of the radiation transmitted and reflected by the dielectric in the forward and reflected directions respectively, and efficient means of applying the external stress signal so that the narrow pass bands of the phase-ordered dielectric have maximum sensitivity to the signal. The narrow pass bands $\Delta\psi_{b,M}$ at the angles $\psi_{b,M}$ of total transmission are defined in the description of the third embodiment of this invention. The pass bands $\Delta\psi_{b,M}$ are a function of the effective lattice spacing and the index of refraction of the phase-ordered dielectric. Any externally applied stress, whether it be mechanical or electrical in nature that alters the lattice spacing or index of refraction, may be considered as a suitable signal for the transducer. The claim of this invention is that the transducing medium is a phase-ordered dielectric which utilizes the narrow pass bands, discovered by the applicant, at small grazing angles of incidence. A preferred embodiment of the invention utilizes a collimated, monochromatic source of optical radiation, a slab of the phase-ordered dielectric mounted in free space, and optical detectors of the radiation transmitted and reflected by the dielectric. Another preferred embodiment of this invention utilizes a discrete, phase-ordered dielectric which is adjoined by an absorbing detector of the transmitted radiation at the exit surface of the dielectric. For both of these embodiments, the thickness of the phase-ordered dielectric does not appreciably affect the transducer properties of the dielectric. The phase-ordered dielectric may be of arbitrary geometry with planar or curved surfaces. However, a slab or disc geometry is preferred.

BRIEF DESCRIPTION OF THE DRAWING

The features of the present invention are better understood from the following descriptions made in conjunction with the accompanying drawing, in which:

FIG. 1a – FIG. 1e are phase-ordered metallic conductors which, in accordance with the principles of the present invention, serve as conductors of metallic radiation guides of conventional modes of propagation;

FIG. 2a is an axial cross-section of a dielectric radiation pipe, made in accordance with the principles of the present invention, for the propagation of electromagnetic waves by successive multiple reflection;

FIGS. 2b and 2c are dielectric radiation pipes, of tubular and slab geometries respectively, made in accordance with the principles of the present invention, for the propagation of electromagnetic waves by successive multiple reflection;

FIGS. 3a and 3b are electromagnetic band pass filters made in accordance with the principles of the present invention;

FIG. 4 is an electromagnetic system, made in accordance with the principles of the present invention, for the investigation of the lattice structure and lattice defects of materials;

FIG. 5 is an electromagnetic mirror made in accordance with the principles of the present invention;

FIG. 6 is an electromagnetic transducer made in accordance with the principles of the present invention;

FIG. 7 shows phase differences across single-scattered wave fronts;

FIG. 8a shows cyclical allowable values of the amplitude delay X in dielectrics;

FIG. 8b shows the cyclical path of the vector X as the angle of incidence is varied from $\pi/2$ to 0 radians;

FIG. 9 shows the intensity of the reflection coefficient in the vicinity of the angles of total transmission;

FIG. 10a shows multiple-scattered waves in a phase-ordered discrete medium;

FIG. 10b shows such waves in a continuous medium;

FIG. 10c shows such waves in a discrete medium with equivalent parameters of a continuous medium;

FIG. 11 shows the intensity of the reflection coefficient for phase-ordered media subject to the conditions

$$V << 1, 0 \leq |\bar{u}| < \infty, Im\ v_{ij} = 0;$$

FIG. 12 shows the intensity of the reflection coefficient in the vicinity of the Bragg angles for total X-ray reflection;

FIGS. 13 and 14 show the effective relative phase velocity and angle of refraction, respectively, in phase-ordered media subject to the conditions set forth above in the description of FIG. 11;

FIG. 15 shows electromagnetic properties of a discrete, phase-ordered metal relative to those of a continuous metal;

FIG. 16 shows the intensity of the reflection coefficient in phase-ordered media subject to the conditions

$$|\bar{u}| << 1, 0 \leq V < \infty, Im\ v_{ij} = 0;$$

FIG. 17 shows the $\bar{u}$–V phase space in phase-ordered media with zero absorption loss;

FIG. 18a shows a semi-unbounded medium j with positive traveling waves;

FIG. 18b shows a semi-unbounded medium i with positive and negative traveling waves; and

FIG. 18c shows a bounded medium j with standing waves.

DETAILED DESCRIPTION OF PREFERRED EMBODIMENTS

Referring now to FIGS. 1a – 1e, the boundary walls of the metallic radiation guides of this invention comprise a phase-ordered metallic conductor 1 which may be of slab, tubular, rod, or other geometry. The conductors 1 of FIGS. 1a – 1e may be arranged to form radiation guides of conventional geometry such as coaxial line, two-wire line, wire over a ground plane, unbalanced and balanced strip line, rectangular waveguide, circular waveguide, etc. For purposes of enhancing the mechanical or electrical properties of the radiation guide, the conductors 1 may adjoin dielectric or metallic media 2 and 3 which are not necessarily phase-ordered. The metallic radiation guides of this invention are not shown in FIGS. 1a – 1e because the geometry of any existing metallic radiation guide with a fixed mode of propagation is applicable to this invention. The claim of this invention is not conductors 1 but rather the use of conductors 1 in existing metallic radiation guides with a defined mode of propagation.

With reference to FIG. 2a – 2c, a phase-ordered dielectric j, which is mechanically supported by a medium 3 and which bounds a dielectric medium i, serves as the reflecting walls of a dielectric radiation guide. The radiation is incident at a grazing angle ψ given by $\psi_{\underline{a},M} \leq \psi \leq \psi_{\bar{a},M}$ where $\psi_{\underline{a},M}$ and $\psi_{\bar{a},M}$ are the angles corresponding to boundaries of total reflection regions at sufficiently small grazing angles of incidence. The radiation propagates in the axial direction in medium i by multiple reflection at the grazing angle ψ as shown in FIG. 2a. Dielectric radiation guides of circular and slab geometries respectively are shown in FIGS. 2b and 2c.

With reference to FIGS. 3a and 3b, a phase-ordered dielectric j serves as an electromagnetic band pass filter at certain small grazing angles of incidence $\psi = \psi_{b,M}$. Except for radiation incident at the angles $\psi_{b,M}$ of total transmission, the incident radiation d is totally or almost totally reflected in the reflected direction e. For $\psi \approx \psi_{b,M}$ (and for the corresponding wavelengths $\lambda \approx \lambda_{b,M}$) the incident radiation a is totally or almost totally transmitted in the forward direction f. In FIG. 3a the radiation d is transmitted in free space to a detector. In FIG. 3b the radiation d is transmitted to an absorbing detector 2 which adjoins the dielectric j.

With reference to FIG. 4, electromagnetic radiation, from a collimated source l, is incident at grazing angles of incidence $\psi \approx \psi_{b,M}$ on a dielectric specimen j whose lattice structure is being investigated. If $\psi \approx \psi_{b,M}$, M = ± 1, ± 2, ... , where $\psi_{b,M}$ are the angles of total transmission for perfect phase order of the specimen j at small grazing angles of incidence, at least some of the incident radiation will be transmitted in the forward direction to a detector 2 even if the specimen does not have perfect phase order. The amount of radiation and the wavelength which are detected by detector 2 is a measure of the lattice defects and lattice spacing of the specimen j.

With reference to FIG. 5, electromagnetic radiation d is incident at a grazing angle of incidence ψ on a phase-ordered dielectric j which is mechanically supported by an adjoining medium 2. If the angle of incidence ψ is given by $\psi_{\underline{a},M} \leq \psi \leq \psi_{\bar{a},M}$ where $\psi_{\underline{a},M}$ and $\psi_{\bar{a},M}$ are the angles corresponding to boundaries of total reflection bands at sufficiently small grazing angles of incidence, then the dielectric j serves as a mirror, with the incident radiation being reflected at the angle ψ in the direction e.

With reference to FIG. 6, electromagnetic radiation d, from a collimated monochromatic source 1, is incident at a grazing angle of incidence $\psi = \psi_{b,M}$ on a phase-ordered dielectric j where $\psi_{b,M}$, $M=\pm 1, \pm 2, \ldots$, are the angles of total transmission at sufficiently small angles of incidence. In the absence of any other signal, the incident radiation d is transmitted by the dielectric j in the forward direction f to a detector 2 with little if any radiation being detected at the angle $\psi_{b,M}$ in the reflected direction e by the detector 3. However, if an external mechanical or electrical signal stress is applied, then the amount of radiation detected by detector 3 in the reflected direction will increase at the expense of the amount of radiation transmitted to detector 2 in the forward direction. The changes in radiation levels at detectors 2 and 3 can be correlated to improve the signal to noise ratio.

There has been described novel systems and components for the utilization of electromagnetic waves in discrete phase-ordered media, particularly in phase-ordered metals at arbitrary angles of incidence and in phase-ordered dielectrics at small grazing angles of incidence. It is evident that those skilled in the art may now make numerous uses and modifications of and departures from the inventive concepts. Consequently, the invention is to be construed as embracing each and every novel feature and novel combination of features present in or possessed by the apparatus and techniques herein disclosed and limited solely by the spirit and scope of the appended claims.

APPENDIX A

ABSTRACT

The multiple-scattering and interference of waves, scattered by each scattering center of the medium, is determined for the entire electromagnetic spectrum by modification of the single-scattering coefficients in Darwin's difference equations. Anomalies, created by the use of Maxwell's field equations for continuous media, are resolved. The phase difference parameters $V = 2\pi d(\sin\psi)/\lambda$ and $\bar{u} = 2\pi d(I - v^2_U)/(2\lambda\sin\psi)$ are introduced to account for the single-scattering phase difference between two successive lattice planes in the reflected and forward directions respectively. (d = lattice spacing parallel to the interface of the incident wavefront, ψ = grazing angle of incidence, λ = incident wavelength, v_U = relative wave phase velocity for continuous media). Whereas Maxwell's field equations are strictly applicable only for the coordinate (0,0) in ($\bar{u}$, V) phase space, the independent, unified electromagnetic theory of this paper is applicable to any point in ($\bar{u}$, V) phase space including the coordinate (0,0). The parameter V was originally introduced by Bragg to explain x-ray reflection but the significance of the parameter $\bar{u}$ has never been reported. A medium is defined to be phase-ordered if $\delta V_{rms} << 1$, $|\delta\bar{u}_{rms}| << 1$ where δV_{rms} and $\delta\bar{u}_{rms}$ are the root-mean-squared deviations from the average value in the medium of the respective parameters. Phase-ordered media therefore include continuous media, perfect crystals with uniform lattice spacing parallel to the interface, amorphous media for which $V_{average} << 1$, $\bar{u}_{average} << 1$, and artificial lattices such as periodic-loaded transmission lines and media. The results are applicable to any wave (electromagnetic, elastic, wave-mechanical De Broglie wave) in phase-ordered media. Semi-unbounded and bounded media are treated.

The effective wave phase velocity in discrete media is dispersive, birefringent, and a periodic function of the single-scattering coefficients. Snell's law of refraction and Fresnel's coefficients of reflection and transmission are a special case of a more general formulation. The critical angle for total reflection and Brewster's angle for total transmission are characteristic of only one of several total reflection and total transmission bands in phase-ordered media. At low grazing angles of incidence, the transmission bandwidth ($\Delta\lambda / \lambda$) $\approx 10^{-11}$, 10^{-5}, and 10^{-3} for microwave, visible, and x-ray wavelengths respectively. For grazing angles of incidence smaller than the critical angle for total reflection, partial or total transmission can occur. This result may be related to the transmission loss in clad optical fibers. The surface resistivity of metallic waveguides and the reflectivity of metallic radiation pipes are significantly less for phase-ordered metals than the values that have been determined on the basis of Maxwell's equations. The theory of the propagation of x-rays, for any arbitrary angle of incidence, is presented. The anomalous x-ray absorption loss at low grazing angles of incidence is resolved.

The theory for bounded media follows easily from the concept of self-consistent, positive travelling, forward and reflected waves at every lattice plane in semi-unbounded media instead of from impedance concepts. The conditions for invisibility, total reflection, and extinction in phase-ordered media are presented. Separate single-scattering coefficients are given for ultrathin films.

The differential wave equation for continuous media and the difference wave equation for phase-ordered media, although not strictly analogous, are identical in the limit of zero lattice spacing. The theory of multiple scattering in phase-ordered media is related to the more general class of problems of radiative transfer including transport phenomena.

1.0 INTRODUCTION

With the notable exception of Darwin's[1] difference equations for Bragg reflection at x-ray wavelengths, the present theory of the propagation of electromagnetic waves in media is based on a second order differential wave equation derivable from Maxwell's[2] field equations. Whereas Darwin's difference equations assume discrete media consisting of equally-spaced lattice planes, Maxwell's field equations are based on the assumption of continuous media. Continuous media are completely characterized by the two tensor parameters, permeability [μ] and permittivity [ϵ], which reduce to scalars for isotropic media. However, classically speaking, it is well known that media are not continuous but are discrete and consist of individual atoms or sub-atomic particles. Each atom is a scattering center of the incident radiation. The resultant transmitted and reflected radiation is determined by the interference of the waves scattered by each scattering center of the medium. If the incident wavelength is very much larger than inter-atomic distances and if the medium consists of enough atoms (or lattice planes) so that the averaging of the discrete scattering events is meaningful, then Maxwell's equations are approximately valid because the discrete medium may then be approximated by a continuum of scattering centers with zero phase difference between the waves scattered by adjacent scattering centers.

As a consequence of approximating discrete media by a continuum, Maxwell's field equations do not directly relate to the single scattering event of each electron in the medium. For example, Snell's[3] law of refraction and Fresnel's[4] coefficients of reflection and transmission at the interface between two unbounded media have never been formally derived from the single scattering of each electron. For the same reason, physical phenomena, such as Brillouin[9] and Raman[10] scattering, which are associated with pertubations or changes in the quantum states of the scattering centers, are also not directly related to Maxwell's equations.

Furthermore, Maxwell's equations are grossly inaccurate whenever the previously stated conditions for approximating a discrete medium by a continuum cannot be satisfied. For example, Maxwell's equations are inadequate for x-rays incident at an arbitrary angle of incidence, for electromagnetic radiation incident at near-zero grazing angles of incidence, and for wave propagation in metals. In these cases, the wavefront phase difference between lattice planes or scattering centers is appreciable. Maxwell's equations can also be grossly inaccurate in the treatment of ultra-thin films and of inhomogeneities in a medium particularly if the number of scattering centers near interfaces is comparable to the total number of scattering centers in the medium.

Therefore, the present state-of-the-art of electromagnetic theory contains several anomalies caused directly by the use of Maxwell's field equations which are based on the assumption of continuous media. The purpose of this paper is to remove these anomalies by the use of a more exact, unified theory based on the assumption of discrete media. The unified aspect of this paper is achieved by determining the interference of the waves scattered by each electron in the medium. In essence, the contribution of this paper is the successful application of Darwin's difference equations to the entire electromagnetic spectrum by suitable modification of the single-scattering coefficients of the equations. For the most part, the theory is not restricted to crystals of long-range order with equal spacing between lattice planes but is also applicable to media consisting of amorphous solids, liquids, gases, and plasma.

In particular, this paper demonstrates that the wave phase velocity in discrete media is not equal to $(\epsilon\lambda)^{-1/2}$ but instead is a periodic function of the single-scattering coefficients. It is shown that the present form of Snell's law of refraction and Fresnel's coefficients of reflection and transmission are a special case of a more exact general formulation. It is shown that the critical angle for total reflection and Brewster's[5] angle for zero reflection are characteristic of only one of several total reflection and total transmission bands in semi-unbounded, discrete media. A theory for bounded, discrete media is developed. The results for discrete media are shown to reduce to those for continuous media as the spacing between scattering centers (or lattice planes) approaches zero.

The mathematics of discrete media differs significantly from that of continuous media. In continuous media, continuous fields are defined at all points of space and are governed by differential field equations[6]. In discrete media, electric and magnetic fields are defined only at points devoid of interacting matter such as in free space or in the space between lattice planes. In discrete media, waves are treated as though they propagate in free space except for scattering by interacting particles. The propagation of waves in discrete media is governed by difference equations.

The difference equations and single-scattering coefficients, which are developed in this paper, can be applied to any wave in much the same manner as the wave equation for continuous media. However, unlike the theory for continuous media, the theory for discrete media is directly related to the physical phenomena associated with the single-scattering event of each particle in the medium. Furthermore, the difference equations leading to the wave equation for discrete media are self-consistent in the sense that the parameter being determined (e.g. the electric field intensity) interacts with itself rather than with another parameter (e.g. the magnetic field intensity). Because of these unique features, the theory of this paper is also expected to provide some unifying aspects to problems concerning elastic waves, wave mechanical De Broglie waves, radiative transfer, and transport phenomena.

2.0 BOUNDARY CONDITIONS AND THE WAVE EQUATION

Qualitative Theory

For waves incident on a semi-unbounded continuous medium (see FIG. 2*b*), the boundary conditions consist of equating the phases of the incident, reflected, and transmitted waves at the interface (contains Snell's laws of reflection and refraction) and of equating the sum of the amplitudes for the tangential components of the incident and reflected waves at the interface to the amplitude of the tangential component of the transmitted wave (results in Fresnel's coefficients of reflection and transmission). However, there are two other boundary conditions which are usually not stated but are always assumed for continuous media. It is assumed that the scattering of waves within the medium is totally constructive for any direction, wavelength, and polarization of the incident radiation. It is also assumed that if one can define or calculate a permeability and permittivity for a medium which extends to all of space, then the same values (or even the concept) of permeability and permittivity are valid for a bounded medium. In a continuous medium, both of these assumptions are correct. The phase difference, between scattering centers in a continuous medium, is zero. The environment at a point near an interface is identical to the environment at a point in the interior of a continuous medium.

For a discrete medium, neither of these assumptions are correct. Nevertheless, if the number of lattice planes (or atoms) near an interface are small compared to the total number in the remainder of the medium, then we shall make the approximation that an effective permeability and permittivity can be defined for a semi-unbounded or bounded medium. However, in discrete media, the interference of scattered waves generally cannot be neglected. Therefore, if a smoothed-out effective permeability and permittivity can be defined for a discrete medium, they must be functions of the direction, wavelength, and polarization of the incident radiation. As the distance between scattering centers approaches zero and the destructive interference of the scattered waves becomes negligible, the effective permeability and permittivity of a discrete medium must reduce to those for a continuous medium. However, the concept of an effective permeability and permittivity in a discrete medium is at best pedagogic and is useful only if one wishes to compare or substitute into existing results for continuous media. The primary parameters which characterize a discrete medium's interaction with waves are the single-scattering coefficients of the medium. Any other parameters which characterize the medium's wave interaction, such as an effective permeability and permittivity, are secondary parameters and are derivable from the single-scattering coefficients.

Before determining the specific single-scattering coefficients for electron scattering of electromagnetic waves (see Section 3.0), it is useful to obtain some preliminary qualitative results of the interference of waves in discrete media. The following statements, which are supported in Section 3.0 are helpful in understanding the qualitative results. If a plane monochromatic wave is incident on a single electron, then the electron scatters a portion of the incident wavefront in all directions. If the same monochromatic wave is incident on an atom containing Z_A bound electrons, then the interference of the waves scattered by each electron is not totally constructive (i.e. is not Z_A times the wave scattered by a single bound electron) but instead is partially destructive because of the non-zero distance between electrons. If the same monochromatic wave is incident on a plane array (lattice plane) of the same atoms, then a portion of the wavefront is transmitted in the forward direction without scattering, a portion is scattered (after undergoing a phase change of $-\pi/2$ radians) in the same forward direction, and the remainder of the wavefront is scattered (after undergoing a phase change of $-\pi/2$ radians) at a grazing angle of reflection equal to the grazing angle of incidence. It is assumed that the lattice plane is of sufficiently large extent so that Fraunhofer aperture diffraction can be neglected. It is assumed that the scattering particles may be in thermal agitation but that diffuse Brillouin scattering can be neglected. It is also assumed that the scattering cross-section of the particles is sufficiently small so that the mutual interactions between scattering centers can be neglected.

If several lattice planes composed of similar scattering particles are placed parallel to the first array at a distance d from it, then interference occurs between the forward and reflected waves from each lattice plane. It will be shown in the quantitative treatment of this section that the interference of the scattered waves from every lattice plane of the medium determine the refractive properties of the medium. However, a qualitative appreciation for the interference of the waves may be achieved if the interference between waves scattered by the atoms of only two adjacent lattice planes is considered. The interference of waves within the atom will also be neglected for the present time.

With reference to FIG. 7, consider that portion AA' of the incident plane wavefront of wavelength λ whose rays are between those which intersect the z axis of the lattice planes at B and B'. The distance BB' $= d$ is the separation between lattice planes. The incident wavefront AA' is of uniform phase normal to the direction ψ of incidence. The reflected wavefront CC', consisting of only the two rays BC and B'C' normal to the direction ψ of reflectance, is not of uniform phase but instead has a phase difference across its wavefront of $2V = (2\pi/\lambda)\,(A'B'C' - ABC) = (2\pi/\lambda)\,(D'B + B'E') = (2\pi/\lambda)\,2D'B = (2\pi/\lambda)\,2d \sin\psi$.

The phase difference V across the wavefront BB' is given by

$$V = (2\pi/\lambda)\,(A'B'C' - ABC/2) = (2\pi d/\lambda)\sin\psi \qquad (2.1)$$

where V is a real number. The wavefront amplitude delay Φ corresponding to the phase difference V is given by

$$\Phi = \exp(-iV) = \exp[-i2\pi d(\sin\psi)/\lambda] \qquad (2.2)$$

Four physical conditions of particular interest, defined by the phase V, are :

$V = 0$ (Condition satisfied by a continuous medium).

$V << 1$ (Condition for most wave-interactions with real media over most of the electromagnetic spectrum).

$V = (2N + 1)(\pi/2)$, N = 1, 2, 3, ... (Condition for total transmission if the refraction of the medium could be neglected and if the interference of waves scattered by the electrons comprising a single atom could be neglected).

$V = N\pi$, N = 1, 2, 3, ... (Condition for Bragg reflection at x-ray wavelengths, neglecting the interference of waves scattered by the electrons comprising a single atom and neglecting the refraction of the medium).

The phase V is tabulated in Table 1 for electromagnetic waves in some solid materials. The three grazing angles of incidence which are considered in Table 1 are

TABLE 1.—THE PHASE V FOR ELECTROMAGNETIC WAVES IN SOLID MATERIALS

[$V = (2\pi d/\lambda)\sin\psi$ (radians)]

	ψ										
	ψ_c (critical angle)	$\psi_{B,N}$ (Bragg angle)	$\pi/2$ (normal incidence)								
	Region										
	X-ray		Ultraviolet		Visible	Infrared		Microwaves			DC
Material	λ (m) far from an absorption edge	$(2d/N)\sin\psi$	10^{-8}	10^{-7}	5×10^{-7}	10^{-6}	10^{-4}	10^{-3}	10^{-2}	10^{0}	∞
Beryllium, α [1]	3.30×10^{-2}	$N\pi$	1.44×10^{-1}	1.44×10^{-2}	2.88×10^{-3}	1.44×10^{-3}	1.44×10^{-5}	1.44×10^{-6}	1.44×10^{-7}	1.44×10^{-9}	0
Aluminum	6.76×10^{-2}	$N\pi$	2.54×10^{-1}	2.54×10^{-2}	5.08×10^{-3}	2.54×10^{-3}	2.54×10^{-5}	2.54×10^{-6}	2.54×10^{-7}	2.54×10^{-9}	0
Nickel	1.06×10^{-1}	$N\pi$	2.22×10^{-1}	2.22×10^{-2}	4.44×10^{-3}	2.22×10^{-3}	2.22×10^{-5}	2.22×10^{-6}	2.22×10^{-7}	2.22×10^{-9}	0
Gold	1.66×10^{-1}	$N\pi$	2.56×10^{-1}	2.56×10^{-2}	5.12×10^{-3}	2.56×10^{-3}	2.56×10^{-5}	2.56×10^{-6}	2.56×10^{-7}	2.56×10^{-9}	0
Quartz, α (SiO_2) [1]	8.25×10^{-2}	$N\pi$	3.08×10^{-1}	3.08×10^{-2}	6.16×10^{-3}	3.08×10^{-3}	3.08×10^{-5}	3.08×10^{-6}	3.08×10^{-7}	3.08×10^{-9}	0

[1] Ordinary form.

$$\psi = \begin{cases} \pi/2 = \text{normal incidence} \\ \psi_{B,N} = \sin^{-1}(N\lambda/2d) = \text{Bragg angles for x-ray reflection, } N = 1, 2, 3, \ldots \\ \psi_c = \sin^{-1}\sqrt{1 - v_{ij}^2} = \text{critical angle for total reflection in continuous media for } v_{ij} \leq 1 \text{ where } v_{ij} = v_i/v_j = \text{relative wave phase velocity for continuous media } i \text{ and } j. \end{cases}$$

It will be noted that $0 < V << 1$ over most of the spectrum. Yet, for this considerable portion of the spectrum, no theory exists except the approximate theory[2-5] based on the assumption $V=0$. At x-ray wavelengths for angles of incidence appreciably larger than the critical angle ψ_c, the phase V satisfies the condition $V > 1$. Yet, for this portion of the spectrum, adequate theory exists only for angles of incidence in the vicinity of the Bragg angles[1,12]. At x-ray wavelengths, for angles of incidence approximately equal to or less than the critical angle ψ_c, the phase V satisfies the condition $0 < V << 1$. Yet, no theory exists for these angles of incidence at x-ray wavelengths except the approximate theory[20] based on the assumption $V=0$. At x-ray wavelengths far from an absorption edge, $\sin \psi_c$ is proportional to the wavelength λ so that the phase $V=2\pi d(\sin \psi_c)/\lambda = V_c$ is independent of wavelength as shown in Table 1.

In addition to the interference of rays scattered in the reflected direction ψ, which is characterized by the phase difference $2V$, the interference of rays scattered in the forward direction must also be considered. With reference to FIG. 7, the distance that a ray travels in the forward direction between adjacent lattice planes is given by $BF = D'F' = d/\sin \psi$. However, the transmitted wavefront FF' normal to the forward direction is not of uniform phase as is the incident wavefront BD'. The rays in the vicinity of BF and D'F' of the incident wavefront BD' are scattered in the forward direction at B and B' respectively whereas the other rays of the wavefront are not scattered. The non-scattered rays undergo a phase change $2\pi d/\lambda\sin\psi$ whereas the scattered rays undergo an additional phase change (of approximately $-\pi/2$ radians as shown in Section 3.0). The average phase difference $2\bar{u}$ of the rays in the wavefront FF' is not known but may be estimated if it is assumed to a first order approximation that half of the rays are not scattered and the other half are uniformly scattered as though they propagated in a continuous medium. If these scattered rays did not change direction, then their phase change in travelling a distance $d/\sin\psi$ would be $2\pi d\, v_{ij}/\lambda \sin \psi$ where $v_{ij} = v_i/v_j$ is their wave phase velocity in medium i relative to that in medium j if media i and j were continuous. However, the rays of a continuous medium are refracted in a direction ψ_j given by $\cos \psi_j = \cos \psi/v_{ij}$. The effective phase change of the scattered rays of the wavefront F'F in the forward direction is therefore $(2\pi d\, v_{ij}/\lambda \sin \psi)(\cos \psi/\cos \psi_j) = 2\pi d\, v_{ij}^2/\lambda\sin\psi$. The average phase difference $2\bar{u}$ of the unscattered and scattered rays in the wavefront F'F is given approximately by $2\bar{u} = 2\pi d(1 - v_{ij}^2)/\lambda \sin\psi$ so that

$$\bar{u} = (2\pi d/\lambda)(1/\sin\psi)(1 - v_{ij}^2/2) \qquad (2.3)$$

The phase $\bar{u}$ is a measure of the refraction caused by a single lattice plane. The phase $\bar{u}$ is related to the phase V by

$$\bar{u} = V(1 - v_{ij}^2)/(2 \sin^2\psi) \qquad (2.4)$$

We shall choose the convention that the angle of incidence ψ is always defined to be a positive angle. Therefore, the phase V is always a positive real quantity whereas the phase $\bar{u}$ may be complex and may assume negative values as well as positive values. The phases $2V$ and $2\bar{u}$ are measures of wave interference in the reflected and forward directions respectively. The phase $2V$ was first used by Bragg to explain reflection at x-ray wavelengths. However, the phase $2\bar{u}$ for wave interference in the forward direction has previously not been reported. Four physical conditions of particular interest, defined by the phase u, are:

$\bar{u} = 0$ (Condition satisfied by a continuous medium).

$|\bar{u}| << 1$ (Condition for most wave interactions with real media over the electromagnetic spectrum).

$|\bar{u}| = (2M + 1)\pi/2$, $M = 0, \pm 1, \pm 2, \pm 3, \ldots$ (Condition for total reflection if multiple scattering between lattice planes and the interference of waves scattered by the electrons comprising a single atom could be neglected).

$\bar{u} = M\pi$, $M = 0, \pm 1, \pm 2, \pm 3, \ldots$ (Condition for total transmission if multiple scattering between lattice planes and the interference of waves scattered by the electrons comprising a single atom could be neglected).

In Table 7, (see p. 62a) the phase $\bar{u}$ is tabulated for free space-metal interfaces for any arbitrary angle of incidence at wavelengths longer than optical wavelengths. For the better conducting metals, $|\bar{u}| > 1$ even for normal incidence. Present theory[2,4] for the propagation in metals is based on the assumption $\bar{u} = 0$. The phase $\bar{u}$ is also tabulated in Table 2 for electromagnetic waves in free space which are incident on quartz at grazing angles of incidence given by

$$\psi = \begin{cases} \pi/2 = \text{normal incidence} \\ \psi_c = \sin^{-1}\sqrt{1 - v_{ij}^2} = \text{critical angle of reflection if } v_{ij} < 1 \\ \psi_{b,M} = \sin^{-1}[d(1 - v_{ij}^2)/M\lambda],\ M = 0, \pm 1, \pm 2, \pm 3, \ldots = \text{qualitative condition for total transmission} \end{cases}$$

It will be noted that $0 < |\bar{u}| << 1$ over most of the electromagnetic spectrum for a free space-quartz interface. Yet, for this considerable portion of the spectrum, no theory exists except the approximate theory[2-5, 20] based on the assumption $\bar{u} = 0$. At grazing angles of incidence sufficiently less than the critical angle ψ_c for total reflection, the phase $\bar{u}$ satisfies the condition $|\bar{u}| > 1$. No adequate theory exists for these grazing angles of incidence.

Fresnel's equations[4], based on the assumption $u = 0$, predict that total reflection should occur for $\psi \leq \psi_c$, v_{ij}^2 4 1, Im $v_{ij} = 0$. However the condition

TABLE 2.—THE PHASE $\bar{U}$ FOR ELECTROMAGNETIC WAVES INCIDENT FROM FREE-SPACE ONTO QUARTZ

[$\bar{u} = (2\pi d/\lambda)(1/2 \sin \psi)(1-v_{ij}^2)$ (radians)]

	Region								
	X-ray	Ultraviolet	Visible	Infrared		Microwaves			DC
ψ	λ (m) far from an absorption edge	1.6×10^{-7}	5×10^{-7}	10^{-6}	10^{-4}	10^{-3}	10^{-2}	10^{0}	∞
$\pi/2$	$4.12\times10^{-2} \sin \psi^c$	-2.9×10^{-2}	-4.0×10^{-3}	-1.9×10^{-3}	-1.9×10^{-5}	-4.3×10^{-6}	-4.3×10^{-7}	-4.3×10^{-9}	0
ψ_c	4.12×10^{-2}	Not applicable, $v_{ij}>1$							
$\psi_{b,M}$	$M\pi$	$M\pi$	$M\pi$	$M\pi$	$M\pi$	$M\pi$	$M\pi$	$M\pi$	0

NOTE.—$d=4.9\times10^{-10}$m, $v_{ij}=1.95$ (microwaves), 1.5 (visible, infrared), 2.0 (ultraviolet).

$\bar{u} = (2M + 1)\pi/2$ which corresponds to totally destructive interference states that total reflection shall occur at grazing angles $\psi_{a,M}$ given by $(1 - v_{ij}^2)/\sin^2\psi_{a,M} = [(2M+1)\pi/V_c]^2$, $M = 0, \pm 1, \pm 2, \pm 3, ...$, Im $v_{ij} = 0$, multiple scattering and the interference of waves scattered by the electrons comprising a single atom are neglected. (2.5)

Equation 2.5 implies that total reflection can occur at discrete angles for $v_{ij} > 1$. Equation 2.5 also implies that total reflection does not occur for all angles less than the critical angle ψ_c for $v_{ij} < 1$.

Even more significant is the condition $\bar{u} = M\pi$, corresponding to totally constructive interference, which states that total transmission should occur for grazing angles $\psi_{b,M}$ given by

$(1 - v_{ij}^2)/\sin^2\psi_{b,M} = (2M\pi/V_c)^2$, $M = 0, \pm 1, \pm 2, \pm 3, ...$, Im $v_{ij} = 0$ (2.6)

multiple scattering and the interference of waves scattered by the electrons comprising a single atom are neglected.

The case $M = 0$ corresponds to the trivial case $v_{ij} = 1$. With reference to Table **2**, total transmission occurs at a grazing angle $\psi_{b,1} \approx 10^{-3}$ radians for visible radiation of wavelength $\lambda = 5 \times 10^{-7}$ m incident on a free space-quartz interface. This result is in contrast to the almost total reflection predicted by Fresnel's equations. With a well-collimated light source and a sufficiently large sample so that Fraunhofer diffraction effects are negligible, total transmission should be able to be observed at approximately this angle. However, it will be shown in Section 4.0 that the width of the transmission band is extremely small and is smaller than the reflection band for Bragg reflection. Whereas Bragg reflection requires materials whose thickness is at least of the order of the penetration depth and whose lattice structure is of corresponding long-range order (x-ray penetration is of the order of 1000 lattice planes), total transmission at low grazing angles of incidence is enhanced as the thickness of the material is reduced because the number of scattering centers is reduced. For total transmission, the lattice structure need only be of short-range order equal to the depth of penetration. If the medium were unbounded, then total transmission would occur only if its lattice structure were of long-range order extending to infinity. Narrow transmission bands at low grazing angles of incidence and total reflection bands for $v_{ij}^2 > 1$ are expected to find application in the investigation of the structure of materials and in the propagation of surface waves, particularly in thin films and fibers for precise filtering and selection of the direction or wavelength of the incident radiation.

The Bragg angle for total reflection at x-ray wavelengths and the angles for total reflection and total transmission given by Equations 2.5 and 2.6 are based on assumptions of the nature of the wavefronts CC' and FF' in FIG. **1**, particularly in regard to the number of scattered and non-scattered rays in each wavefront. In Sections 3.0 and 4.0 an independent, exact, quantitative treatment, free of such assumptions, gives results which are in substantial agreement with the preceding qualitative results.

The phases V and $\bar{u}$ are both functions of the lattice spacing d parallel to the interface. However, most materials are polycrystalline or amorphous and do not have a well-defined lattice structure of long-range order. Furthermore, many crystals do not have a lattice structure in which the atoms are arranged in equally spaced planes parallel to the interface and therefore cannot be characterized by a single lattice spacing d. Ewald[8] has circumvented this latter difficulty by introducing the concept of reciprocal vectors. The problem of treating polycrystalline and amorphous materials has remained unsolved and has been a severe limitation in many other fields of physics besides electromagnetic theory. The following method for treating polycrystalline and amorphous materials is now offered.

The interference of secondary waves which are multiple-scattered is generally dependent upon the order in which they interfere and the number of scattering events. For these reasons the effective phase interference parameters V and $\bar{u}$ of a polycrystalline or amorphous medium generally are not equal to the average phase parameters $V_{av} = 2\pi d_{av}(\sin\psi)/\lambda$ and $\bar{u}_{av} = 2\pi d_{av}(1 - v_{ij}^2)/(2\lambda\sin\psi)$ where d_{av} = the average spacing of the lattice planes parallel to the interface in a polycrystalline material or the average spacing between neighboring atoms in an amorphous material. However, there are conditions for which the resultant interference in polycrystalline and amorphous materials is relatively unaffected by the use of the average phase interference parameters V_{av} and $\bar{u}_{av}$. For example, if the spacing d_{mn} for each pair (m,n) of adjacent lattice planes in a polycrystalline material or of atoms in an amorphous material satisfied the conditons $|V_{mn} - V_{av}| << 1$, $|\bar{u}_{mn} - \bar{u}_{av}| << 1$, then $V \approx V_{av}$ and $\bar{u} \approx \bar{u}_{av}$ because the interference of the phase differences are constructive. However even if a small percentage of the spacings d_{mn} do not satisfy the above conditions, the resultant interference is still relatively unaffected assuming that the scattering cross-sections of the atoms associated with the spacings d_{mn} are not appreciably larger than the majority of the scattering centers. Therefore let the root-mean-squared deviation from the mean of the parameters d, V, and ū be given respectively by $\delta d_{rms} = \sqrt{(d_{av} - d_{mn})^2}$, $\delta V_{rms} = \sqrt{(V_{av} - V_{mn})^2}$, $\delta\bar{u}_{rms} = \sqrt{(\bar{u}_{av} - \bar{u}_{mn})^2}$. The effective phase difference parameters in polycrystalline and amorphous materials may therefore be estimated to a first order of approximation, subject to the appropriate conditions, by

$$V \approx V_{av},\ \delta V_{rms} << 1$$
$$\bar{u} \approx \bar{u}_{av},\ |\delta\bar{u}_{rms}| << 1 \quad (2.7)$$

A material which satisfies Equations 2.7 is defined as "phase-ordered." If Conditions 2.7 are satisfied by polycrystalline or amorphous materials, the problem of obtaining approximately valid results with such materials reduces to the problem of treating a single crystal for any value of the phase parameters V and $\bar{u}$.

Before concluding the qualitative theory, it is useful to state some properties regarding the phase order of materials. Continuous media and perfect single crystals, whose atoms are arranged in equally spaced planes parallel to the interface, satisfy the conditions $\delta d_{rms} = \delta V_{rms} = \delta\bar{u}_{rms} = 0$. Wave interactions, in materials which satisfy the conditions $V_{av} << 1$, $\bar{u}_{av} << 1$, also satisfy the conditions $\delta V_{rms} << 1$, $|\delta\bar{u}_{rms}| << 1$ but the converse is not true. For a solid amorphous material, the average spacing d_{av} between neighboring atoms is determined by assuming that each atom (or molecule) occupies a volume $(4\pi/3)(d_{av}/2)^3 = A/\rho N_o$ where A = atomic (or molecular) weight (gms), ρ = material density (gm/cm^3), and N_o = Avogadro's number = 6.022×10^{23}. In Table 3 the structured vs. amorphous spacings of some solid materials are compared. Most liquids can be treated as amorphous materials. However, some liquid solutions should be treated as polycrystalline materials since they have a well-defined lattice structure of short-range order with an average lattice spacing which can be comparable to those for solids or which can be more than an order of magnitude larger. Gases and plasmas are amorphous media whose average time-independent lattice spacing is equal to the average distance between scattering centers. A property common to all media is that the scattering centers are in thermal motion. For solids, the root-mean-squared thermal fluctuation $\sqrt{\overline{\delta_T^2}}$ of the lattice is small compared to the average lattice spacing. For gases,

TABLE 3.—STRUCTURED VS. AMORPHOUS SPACINGS OF SOME SOLID MATERIALS

Material	Structured spacings at temperature T=293° K. Closest approach (10^{-10}m.)	Lattice parameter a (10^{-10}m.)	Amorphous spacing (average spacing between atoms) (10^{-10}m.)
Beryllium, α [1]	2.225	2.285	2.48
Aluminum	2.862	4.049	3.16
Nickel	2.491	3.524	2.76
Gold	2.884	4.078	3.195
Quartz, α (SiO_2) [1]		4.9	4.15
Silver	2.888	4.086	3.20
Copper	2.556	3.615	2.84
Tungsten, α [1]	2.739	3.156	3.12
Lead	3.499	4.950	3.86
Mercury	[2] 3.006	[2] 2.006	3.56

[1] Ordinary form.
[2] T=227° K.

the root-mean-squared fluctuation $\sqrt{\overline{\lambda_T^2}}$ of the molecules is equal to the average spacing between molecules. Therefore, for gases, $\sqrt{\overline{\delta_T^2}} = \delta d_{rms} = d_{av}$, $\delta V_{rms} = V_{av}$, and $\delta \overline{u}_{rms} = \overline{u}_{av}$. In any medium, pertubations or changes in the quantum state of the scattering centers can cause diffuse scattering and also modification of the specularly scattered radiation. The diffusely scattered radiation resulting from thermal agitation has been treated by Brillouin[9] and that resulting from higher order acoustical vibrations and rotations has been treated by Raman[10]. The effect of thermal agitation on the specularly scattered radiation is well-established for single crystals[11] and is usually treated as a modification of the atomic structure factor (see Section **3.0**). However, as illustrated above for gases, an additional effect of thermal agitation is to decrease the phase order of the medium because the effective values of δd_{rms}, δV_{rms}, and $\delta \overline{u}_{rms}$ are increased.

Quantitative Theory

Thus far, only the interference of single-scattered waves by two adjacent planes has been considered and qualitatively described by the phase parameters V and $\overline{u}$. The quantitative theory, which now follows, determines the interference of waves multiple-scattered by all the lattice planes of the medium.

The author has found that Darwin's difference equations are applicable to the entire electromagnetic spectrum if the single-scattering coefficients of the equations are suitably modified by removing the approximations which Darwin made for the condition $V = N\pi$ at x-ray wavelengths. For purposes of comparison, the notation of this paper corresponds approximately to that of Compton and Allison[12] who present a summary of Darwin's work and other related work. With reference to FIG. **10***a*, consider a plane monochromatic wave of wavelength λ and amplitude T_1 which is incident from medium *i* at a grazing angle of incidence ψ on a semi-unbounded medium *j*,consisting of lattice planes of separation *d* measured normal to the interface, that extends to $\pm\infty$ in the *x* and *y* directions and from 0 to $+\infty$ in the *z* direction. For purposes of normalization, it will be assumed that $T_1 = 1$ and is the electric field intensity, on the *z* axis just above the lattice plane $r = 1$, polarized either normal or parallel to the plane of incidence. In FIG. **10***a*, the direction of the electric vector T_1 is not shown and only the direction of propagation of T_1 is shown. The medium is extended to $\pm\infty$ in the *x* and *y* directions so that Fraunhofer aperture diffraction need not be considered. The following conventions will be adopted. If the lattice plane nearest the interface is labeled $r = 1$, as in FIG. **10***a*, then the amplitudes of the total forward and reflected waves, measured on the *z* axis just above the *r* th lattice plane, are T_r and S_r respectively. (Compton and Allison use the notation T_{r-1} and S_{r-1}.) The total amplitudes T_r and S_r at each lattice plane are the steady-state resultant amplitudes caused by the interference of the waves scattered by each lattice plane of the medium. Medium *i*, from which the radiation of wavelength λ is incident, need not be restricted to vacuum but can be any medium i with its own lattice structure since T_1 and S_1 are the resultant amplitudes and include the effect of any scattering in medium *i*. If the wavelength of the incident radiation in medium *i* is λ_i, then $\lambda_i = \lambda$. In defining the total forward wave at each lattice plane, it is convenient to set its grazing angle of incidence at each lattice plane equal to the grazing angle of incidence ψ at the first lattice plane. The actual direction of travel of the total forward wavefront is found by comparing complex amplitudes of the total forward wave at each lattice plane. It will be shown in Section 3.0 that the grazing angle of reflection of the total reflected wave at each lattice plane is equal to ψ. For unbounded continuous media (see FIG. **10***b*), the boundary conditions are defined only at the interface and are in terms of the incident amplitude $T = T_1 = 1$, the reflected amplitude S, and the transmitted amplitude $\vec{\tau} = \vec{T} + \vec{S}$. For unbounded discrete media, the reflected and transmitted waves are determined by the boundary geometry of FIG. **10***a*. Nevertheless, the results for discrete media can be used to define an equivalent set of boundary conditions (see FIG. **10**c) at the interface in terms of an incident amplitude $T' = T_1 = 1$, a reflected amplitude $S' = S_1$, and a transmitted amplitude $\vec{\tau_1} = \vec{T_1} + \vec{S_1}$. The direction ψ_j', phase velocity v_j', and amplitude τ_1 of the transmitted wave in a discrete medium j reduce to the corresponding parameters ψ_j, v_j, and τ of a continuous medium as the lattice spacing *d* approaches zero.

The single-scattered specular waves at each lattice plane, which were qualitatively described earlier in this section, are not shown in FIG. **10***a*. However, they relate the total forward and reflected waves at a given lattice plane to the total forward and reflected waves at each of the immediately adjacent lattice planes. By definition, let

a = single-scattering coefficient of reflection = amplitude of the single-scattered reflected wave for unit amplitude of the incident wave

t = single-scattering coefficient of transmission = resultant amplitude of the single-scattered and non-scattered forward waves for unit amplitude of the incident wave.

The single-scattering coefficients *a* and *t* are generally a function of the polarization (normal or parallel) and nature (electric or magnetic field intensity) of the incident wave. The single-scattering coefficients are dimensionless and should not be confused with a scattering cross-section which has the dimensions of area.

The normalized total amplitudes S_r/T_1 and T_r/T_1 are, in effect, multiple-scattering coefficients because they are the resultant normalized amplitudes after multiple scattering has occurred. With reference to FIG. **10***a*, the total amplitudes S_r and T_r are related to the single-scattering coefficients by[12]

$$S_r = a\,T_r + t\Phi\,S_{r+1}$$
$$T_r = t\Phi\,T_{r-1} + a\Phi^2\,S_r \tag{2.8}$$

where Φ is given by Equation 2.2. Equations 2.8 are similar in form to those derived by Darwin with the exception that the single-scattering coefficients a and t are not necessarily equal to those of Darwin. Equations 2.8, together with the equations which specify the single-scattering coefficients a and t (see Section 3.0), comprise the interference and the boundary conditions of steady-state waves in phase-ordered media. From Equation 2.8b, $S_r = (T_r - t\,\Phi\,T_{r-1})/a\Phi^2$ and $S_{r+1} = (T_{r+1} - t\,\Phi\,T_r)/a\Phi^2$. Substitution of these expressions into Equation 2.8a yields Darwin's second-order difference equation

$$T_{r+1} - 2\,[1 + \Phi^2(t^2 - a^2)/2\,\Phi\,t]T_r + T_{r-1} = 0 \tag{2.9}$$

An identical equation in terms of the total reflected waves S_{r+1}, S_r, and S_{r-1} also follows from Equation 2.8. Equation 2.9 is the wave equation for steady-state waves in phase-ordered media.

Equation 2.9 is a homogeneous, second-order difference equation of the form $T_{r+1} - 2A\,T_r + T_{r-1} = 0$ where A is a constant given by

$$A = [1 + \Phi^2(t^2 - a^2)/2\,\Phi\,t] \tag{2.10}$$

Equation 2.9 has the general solution[13]

$$T_r = c_1\,X^{r-1} + c_2\,X^{-(r-1)} \tag{2.11}$$

where

$X = A \pm \sqrt{A^2 - 1} = A \pm i\sqrt{1 - A^2} = \exp(\pm i\theta)$, $\theta = \cos^{-1}A$

c_1, c_2 = constants which must satisfy boundary conditions. As a corollary of Equation 2.11, the parameter A is given by

$$A = X^{-1} + X = [1 + \Phi^2(t^2 - a^2)/2\,\Phi\,t] \tag{2.11a}$$

Conservation of energy requires that $|T_r| \leq |T_1|$. For a wave of amplitude T_1 incident ar $r = 1$ on a semi-unbounded medium ($1 \leq r < \infty$), the wave propagates only in the $+r$ direction. For such a case, Equation 2.11 reduces to

$$T_r = T_1\,X^{r-1},\ X = A \pm \sqrt{A^2 - 1},\ |X| \leq 1 \tag{2.12}$$

where A is given by Equation 2.10. Similarly, the wave equation for the reflected wave is given by $S_{r+1} - 2A\,S_r + S_{r-1} = 0$ which has the solution

$$S_r = S_1\,X^{r-1},\ X = A \pm \sqrt{A^2 - 1},\ |X| \leq 1 \tag{2.13}$$

The phase of the reflected wave increases at the same rate and in the same direction as the forward wave. Substitution of Equation 2.13 into Equation 2.8a yields the equations $S_r = a\,T_r + t\,\Phi\,S_r\,X$ and $S_1 = a\,T_1 + t\,\Phi\,S_1\,X$. The amplitudes S_1 and S_r are therefore given by

$$S_1/T_1 = S_r/T_r = a/(1 - \Phi\,X\,t)\ \text{, semi-unbounded media} \tag{2.14}$$

With the use of Equation 2.11a, it can be shown that

$$a/(1 - \Phi\,X\,t) = (1 - \Phi t/X)\,/a\Phi^2 \tag{2.14a}$$

Equation 2.14 (for S_1/T_1) was first derived by Darwin[12] to explain Bragg reflection at x-ray wavelengths. However it will be shown in Sections 3.0 and 4.0 that Equation 2.14 gives the multiple-scattering reflection coefficient in phase-ordered, semi-unbounded media over the entire electromagnetic spectrum if the single-scattering coefficients a and t are suitably derived.

However, without explicitly deriving the single-scattering coefficients a and t, it is possible to express the secondary parameters, which characterize the refractive parameters of a medium, as a function of a and t. The parameter X, given by Equation 2.12, is the multiple-scattering coefficient of amplitude delay which may be written in the more familar form

$$X = \exp[-\gamma_{jz}'\,z/(r-1)] = \exp(-\gamma_{jz}'\,d) \tag{2.15}$$

where

$\gamma_{jz}' = \alpha_{jz}' + i\,\beta_{jz}'$ = effective propagation constant in the direction of the z axis in the medium j.

α_{jz}' = effective attenuation constant in the direction of the z axis in the medium j.

β_{jz}' = effective phase constant in the direction of the z axis in the medium j.

Therefore,

$$\gamma_{jz}' = -(1/d)\,ln\,X \tag{2.16}$$

where X is given by Equation 2.12. The effective wavelength λ_z' in the direction of the z axis in medium j is given by

$$\lambda_{jz}' = 2\pi/Im(\gamma_{jz}') = 2\pi/\,\beta_{jz}' \tag{2.17}$$

With reference to FIG.2c, the effective direction ψ_j' of wave propagation and the effective relative wave phase velocity $v_{ij}' = v_i/v_j'$ are the refractive parameters of medium j. Snell's law of refraction (see Equation 4.22 where the law is derived) may be used to give

$$\sin\psi_j' = [1 - (\cos^2\psi)/(v_{ij}')^2]^{1/2} \tag{2.18}$$

The effective propagation constant γ_j' in the direction normal to constant phase front in medium j is given by

$$\gamma_j' = \gamma_{jz}'/\sin\psi_j' = i\,2\pi v_{ij}'/\lambda = \alpha_j' + i\,\beta_j' \tag{2.19}$$

The effective wavelength λ_j' in the direction normal to constant phase front in medium j is given by

$$\lambda_j' = 2\pi\,/\,\beta_j' = \lambda/v_{ij}',\ \text{Im}\ v_{ij}' = 0 \tag{2.20}$$

Solving Equations 2.16, 2.18, and 2.19 for v_{ij}' and ψ_j',

$$v_{ij}' = v_i/v_j' = [\cos^2\psi - (\lambda\,ln\,X/2\,\pi\,d)^2]^{1/2} \tag{2.21}$$

$$\cos\psi_j' = \cos\psi/v_{ij}' = [1 - (\lambda\,\sec\psi\,ln\,X/2\pi\,d)^2]^{-1/2} \tag{2.22}$$

where X is given by Equation 2.12.

Equations 2.15 – 2.22 relate the refractive properties of a medium to the single-scattering coefficients of the medium. Additional secondary parameters are the effective relative permittivity $\epsilon_j'/\,\epsilon_i$ and the effective relative permeability μ_j'/μ_i. If the magnetic properties of the adjacent media are identical, then $\mu_j'/\mu_i = 1$ and

$$\epsilon_j'/\epsilon_i = (\nu_{ij}')^2 = [\cos^2\psi - (\lambda \ln X/2\pi d)^2]^{-1} \tag{2.23}$$

where the incident amplitude T_1 (see FIG. 10) refers to the electric field intensity. If the non-magnetic electrical properties of adjacent media are identical so that $\epsilon_j'/\epsilon_i = 1$, then T_1 refers to the amplitude of the magnetic field intensity and μ_j'/μ_i is given by Equation 2.23. If $\mu_j/\mu_i \neq 1$ and $\epsilon_j/\epsilon_i \neq 1$, then Equations 2.8 must be supplemented by additional field relationships which specify the interaction between the electric and magnetic properties of the media. In this paper, it will be assumed that the magnetic properties of adjacent media are identical to that of free space and that T_1 refers to the incident electric field intensity.

With reference to FIG. **10***c*, the resultant amplitude τ_r of the total forward wave T_r and the total reflected wave S_r at the lattice plane r is given by

$$\vec{\tau}_r = \vec{T}_r + \vec{S}_r = (\vec{T}_1 + \vec{S}_1) X^{r-1}$$
$$= \begin{cases} (\tau_r)_n = [(T_1)_n + (S_1)_n] X_n^{r-1} \\ (\tau_r)_p = \dfrac{\sin\psi}{\sin(\psi_j)_p} [(T_1)_p + (S_1)_p] X_p^{r-1} \end{cases} \tag{2.24}$$

where the subscripts n and p denote whether the incident electric field intensity is polarized normal or parallel respectively to the plane of incidence. For normal polarization, the electric vectors $\vec{S}_1$ and $\vec{T}_1$ are in the same direction so that their resultant $\vec{\tau}_1$ is simply the sum of the two amplitudes. For parallel polarization, the electric vectors $\vec{S}_1$, $\vec{T}_1$, and their resultant $\vec{\tau}_1$ are not in the same direction but their directions are related to each other by the angles ψ and $(\psi_j')_p$ defined in FIG. **2***c*. The multiple-scattering coefficients of reflection S_1/T_1 and transmission τ_1/T_1 at the interface of semi-unbounded media are therefore given by

$$S_1/T_1 = \begin{cases} (S_1/T_1)_n = [a/(1-\Phi Xt)]_n \\ (S_1/T_1)_p = [a/(1-\Phi Xt)]_p \end{cases}$$
$$\tau_1/T_1 = \begin{cases} (\tau_1/T_1)_n = 1 + (S_1/T_1)_n \\ (\tau_1/T_1)_p = [\sin\psi/\sin(\psi_j)_p][1 + (S_1/T_1)_p] \end{cases} \tag{2.25}$$

As the lattice spacing d approaches zero, Equations 2.25 must reduce to Fresnel's coefficients of reflection S/T and transmission τ/T for continuous media.

Equations 2.10 – 2.14, which completely determine the properties of waves in phase-ordered media, were derived independently of Maxwell's field equations. Equations 2.24 and 2.25 are derived from Equations 2.10 – 2.14. Equations 2.15 – 2.23, which determine the effective smoothed-out secondary parameters as functions of the single-scattering coefficients a and t, enter the discussion of the propagation of waves in media only if the equivalent refractive properties of the medium are to be determined. It should be understood however that refraction is nothing more than an effective smoothed-out parameter interpretation of the interference of waves scattered by each lattice plane. In Section 4.0, the refractive properties of media are derived directly from Equations 2.10 – 2.14.

3.0 SINGLE-SCATTERING COEFFICIENTS

The single-scattering coefficients a and t specify the amplitudes of the reflected and transmitted waves respectively which result when a plane, monochromatic wave, whose electric field intensity is of unit amplitude $T_1 = 1$, is incident on either a plane array of atoms in free space or on a lattice plane of a phase-ordered medium. The coefficients a and t are a function of the polarization of the incident wave. The subscripts n or p will be used to denote whether the incident electric field intensity is polarized normal or parallel respectively to the plane of incidence. The coefficients a and t are determined by considering in progressive order, the scattering by:

1. Free electrons
2. Bound electrons in an atom, molecule, or unit cell of a crystal
3. Bound electrons of a plane array in free space
4. Bound electrons of a lattice plane in a phase-ordered medium.

Scattering by Free Electrons

Consider a plane wave of wavelength λ, whose electric field intensity is of amplitude $T_1 = 1$, which is incident at time $t = 0$ on the origin O in the forward direction at an angle $\phi = 0$ with respect to the x axis. Assume that an electron of charge $-e$, mass m, and zero spin, is located at the origin O and is not subject to any forces of constraint. A portion of the incident wavefront is scattered by the electron in all directions as a transverse electromagnetic wave. If the recoil of the electron is neglected, then the scattered wave $E(\rho, \phi, t)$ at a distance ρ from the origin O at a field point $P(\rho,\phi)$ is given (after J. J. Thomson) by[14] $E(\rho, \phi, t) = E(\rho, \phi) \exp[i\, 2\pi(ct-\rho)/\lambda]$ where

$$E(\rho,\varphi) = \begin{cases} E_n = -e^2(\text{sign}\cos\varphi)/4\pi\epsilon_0\rho mc^2,\ T_1 = T_{1n} = 1 \\ \quad \text{is normal to the plane defined by OP and the x axis.} \\ E_p = -e^2\cos\varphi/4\pi\epsilon_0\rho mc^2,\ T_1 = T_{1p} = 1 \\ \quad \text{is parallel to the plane defined by OP and the x axis} \end{cases} \tag{3.1}$$

$$\text{sign}\cos\varphi = \begin{cases} 1, & 0 \le |\varphi| < \pi/2 \\ -1, & \pi/2 < |\varphi| \le \pi \\ 0, & \varphi = \pi/2 \end{cases}$$

Equation 3.1 is expressed in rationalized MKS units where ϵ_0 is the permittivity of free space. The minus sign enters into Equation 3.1 because the acceleration of the electron is $-e\,T_1/m$. The quantity, sign cos ϕ, is necessary in Equation 3.1*a* because $E_n = E_p$ for $\phi = 0$ and $\phi = \pi$ radians. The angular distribution of the scattered radiation of Equation 3.1 is known as Rayleigh scattering. The total power P_s scattered in all directions is found by integrating the intensity E^2 of the scattered wave over the surface of a sphere of radius ρ and is given by[14]

$$P_s/T_1^2 = \int_0^\pi (E^2/T_1^2)\, 2\pi\rho \sin\varphi\rho\, d\varphi = e^4/6\pi\epsilon_0^2 m^2 c^4 = 32\pi r_0^2/3 \tag{3.2}$$

where r_o = classical radius of the electron = 2.818×10^{-15}m. The quantity P_s/T_i^2 has the dimensions of area and is defined as the scattering cross-section of a free electron. If the incident wavefront is of unit area, then Equation 3.2 gives the fraction of the incident wavefront which is scattered. The wave scattered by a free electron in the forward direction is 180 degrees out of phase with the incident wave and the power scattered in any given direction ϕ (of zero cone angle) is zero. Therefore, the amplitude t of the total specularly transmitted wave in the forward direction is $t = (1 - P_s/T_1^2)^{1/2}$ and the amplitude a of the specularly reflected wave in a direction ϕ from the forward direction is $a = 0$ where t and a are normalized with respect to T_1.

Scattering by Bound Electrons in an Atom, Molecule, or Unit Cell of a Crystal

Electrons are generally not free from forces of constraint and, for example, may be bound to the nucleus of an atom. Consider an atom of Z_A electrons which are quasi-elastically bound to the nucleus. If the interference of the waves scattered by each electron were totally constructive, then the amplitude at $P(\rho, \phi)$ of the wave scattered by the k th bound electron would be given by[15] $E(\rho, \phi)\ [1 - \omega_k^2/\omega^2 - i\ 2\ \alpha_k/\omega]^{-1}$ where

$\omega = 2\pi c/\lambda$ = radian frequency of the incident radiation

ω_k = resonant radian frequency of the k th bound electron

α_k = damping factor of the k th bound electron.

However if the wavelength of the incident radiation is comparable to the distance between the bound electrons, then the interference of each scattered wave is not totally constructive. The atomic structure or form factor[16] $F(\phi/2, h)$ has therefore been defined and tabulated to account for the interference of waves scattered by each electron in an atom, molecule, or unit cell of a crystal, assumed to be at rest at absolute temperature T=0. The parameter $h = (4\pi / \lambda) \sin (\phi/2)$. The quantity $F(\phi/2, h)$ is a positive real number with the following properties:

$F(0, h) = F(0, 0) = Z_A$ = number of bound electrons

$F(\phi/2, 0) = Z_A$

$0 \leq F(\phi/2, h) \leq Z_A$

$F(\phi/2, h) = Z_A^{1/2}$ if the electrons scatter incoherently.

Accordingly, the resultant amplitude f/ρ of the scattered wave at $P(\rho, \phi)$, caused by scattering and interference of all the bound electrons in the atom is given by

$$f/\rho = E(\rho, \varphi) F(\varphi/2, h)(1/Z_A) \sum_{k=1}^{k=Z_A} [1 - (\omega_k/\omega)^2 - i(2\alpha_k/\omega)]^{-1} \quad (3.3)$$

If $\omega << \omega_k$ and $\omega << \alpha_k$, then the forces of constraint are approximately zero for each electron and Equation 3.3 reduces to

$$f/\rho = E \cdot F(\phi/2, h)\ ;\ \omega << \omega_k,\ \omega << \alpha_k \quad (3.4)$$

An electron need not be bound to a single atom but may also be bound to more than one atom as in a molecule or unit cell of a crystal. Therefore, in addition to the electronic polarization of an atom, the incident radiation can cause atomic and orientation polarizations of the molecule[17]. If the incident wavelength is large compared with the dimension of the scattering particle or the scattering dipole, then regardless of the particular mechanism which gives rise to the scattering, the scattered radiation will be characterized by a Rayleigh angular distribution (see Equation 3.1). If the atoms comprising the molecule or unit cell are in thermal agitation at an absolute temperature T > 0, with a mean square displacement $\overline{\delta_T^2}$ of the distances between them, then there is additional destructive interference and the corresponding atomic structure factor $F_T(\phi/2, h)$ is given by[11]

$$F_T(\phi/2, h) = F(\phi/2, h) \exp [-M_T(\phi/2, h)] \quad (3.5)$$

where

$\exp [-M_T(\phi/2, h)]$ = atomic structure temperature factor = $\exp [-8\pi^2\overline{\delta_T^2}(\sin^2\phi/2)/\lambda^2]$ (after Waller and James[11])

It should be noted that $F_T(0, h) = F(0, h)$. For a plane wave of electric field intensity $T_1 = 1$, which is incident in the x direction ($\phi = 0$) and which is scattered by bound electrons at the origin O, the amplitude f/ρ of the scattered wave at $P(\rho,\phi)$ can be written in the general form

$$f/\rho = \begin{cases} f_n/\rho = (G_n/\rho)(\text{sign}\cos\varphi)F_T(\varphi/2, h)\quad T_1 = 1 \\ \text{is normal to the plane defined by OP and the } x \text{ axis} \\ f_p/\rho =)G_n/\rho)(\cos\varphi)F_T(\varphi/2, h),\ T_1 = 1 \\ \text{is paralled to the plane defined by OP and the } x \text{ axis} \end{cases} \quad (3.6)$$

where G_n (volts) is independent of the direction ϕ, distance ρ, and interference of the scattered waves. For example, for electronic polarization of the atom,

$$G_n = (-e^2/4\pi\epsilon_o mc^2 Z_A) \sum_{k=1}^{k=Z_A} [1 - \omega_k^2/\omega^2 - i2\alpha_k/\omega]^{-1} \text{ volts.}$$

Scattering by Bound Electrons of a Plane Array in Free Space

Consider now M_o scattering centers (atoms or molecules) per unit area arrayed in a single plane of infinite extent ($x = y = \pm\infty$) *in free space at* $z = 0$. Let a plane wave, of electric field intensity $T_1 = 1$ and wavelength λ. be incident on the plane array at a grazing angle ψ. We shall assume that the spacing d' between scattering centers is sufficiently large so that the electrons of a scattering center scatter the incident wavefront independently of the electrons in an adjacent scattering center. This condition is equivalent to the requirement that the radius of the electron scattering cross-section be much smaller than the spacing $d'/2$ projected normal to the incident wavefront. From Equation 3.2 we therefore have the condition

$$r_o\ (32/3)^{1/2} << (d'/2) \sin \psi,\ \text{electronic polarization of the atom},\ \omega << \omega_k \quad (3.7)$$

From Table 3, it is noted that $d' > 2.2 \times 10^{-10}$m. The atoms can therefore be assumed to scatter independently of one another provided that the grazing angle of incidence $\psi >> 6 \times 10^{-5}$ radians.

Darwin has shown[12] that if the plane array is of infinite extent and the scattering centers scatter independently of one another, then, if higher order terms are neglected, the interference of the individual scattered waves results in a specular scattered wave $(-i\sigma)\exp (i2\pi/\lambda)(ct - \rho)$ in the forward direction and a specular scattered wave $(-i\,s)\exp (i2\pi/\lambda)(ct - \rho)$ in a direction $\phi = 2\,\psi$ from the forward direction. The distance ρ is measured from the field point P to the origin of the plane array. The direction $\phi = 2\psi$ corresponds to a grazing angle ψ of reflection equal to the grazing angle ψ of incidence. According to Darwin, σ and s are given by

$$\sigma \approx G_n F(\mathbf{0}, h)(\text{sign} \cos \mathbf{0})\ M_o\lambda/\sin \psi = G_n\ F(\mathbf{0}, h)\ M_o\lambda/\sin \psi,\ |\sigma| << 1 \tag{3.8}$$

$$s = \begin{cases} s_n \approx [F_T(\psi, h)/F(0, h)]\delta(\text{sign} \cos 2\psi),\ |\delta|\mu_1,\ T_1 = 1 \\ \text{is normal to the plane of incidence} \\ s_p \approx [F_T(\psi, h)/F(0, h)]\delta \cos 2\psi,\ |\delta|\mu 1,\ T_1 = 1 \\ \text{is parallel to the plane of incidence} \end{cases} \tag{3.9}$$

where

$\psi = \phi/2$

G_n is defined by Equation 3.6.

The condition $|\sigma| << 1$ was not explicitly stated by Darwin but was implied since $\sigma \rightarrow \infty$ as $\psi \rightarrow 0$. Conservation of energy requires that $|\sigma| \leq 1$. The amplitude a of the reflected wave in a direction $2\ \psi$ from the forward direction and the amplitude t of the total transmitted wave in the forward direction were given by Darwin as $a = -i\ s$ and $t = 1 + (-i\sigma)$. The author has found that retention of the higher order terms, which Darwin neglected in his Taylor series expansion for ρ_A (ρ_A is the distance from an individual scattering atom to the field point P), results in a correction factor for the scattered forward and reflected waves of exp $[(i2\pi\rho/\lambda)(\lambda g/\rho \sin \psi)^2]$ where g is an unspecified functic of the angle of incidence ψ. The scattered waves therefore propagate with a velocity which is different from the velocity in free space. Another problem with Darwin's expressions for a and t is that t clearly violates the requirement for conservation of energy if σ is real. It has been arugued[12] that σ is always complex and $|t| \leq 1$ because the scattering process is associated with an absorption loss. This argument is clearly fallacious at wavelengths far removed from an absorption band. In an effort to remove these difficulties of Darwin's expressions for a and t, let

$$a = a_o \exp\ (i\ u) = -i\ s \exp\ (i\ u) \tag{3.10}$$

$$t = t_o \exp\ (i\ u)$$

where

$a_o = -i\ s,\ |a_o| \leq 1$

t_o = complex amplitude of the total wave transmitted in the forward direction if there were no refraction caused by a single plane array in free space; $|t_o| \leq 1$.

u = complex phase change, of the reflected and total forward wave, resulting from the refraction introduced by a single plane array in free space.

Conservation of energy requires that $|a|^2 + |t|^2 \leq 1$. A more convenient expression for the conservation of energy is obtained by the use of the complex energy amplitudes a_o^2 and t_o^2. If it is assumed that t_o is $\pi/2$ radians out of phase with a_o, then

$$t_o^2 - a_o^2 = 1 \tag{3.11}$$

It immediately follows that

$$t_o = (1 - s^2)^{1/2} \tag{3.12}$$

If σ is real, the total wave in the forward direction consists of an unscattered wave and a scattered wave which are $\pi/2$ radians out of phase with each other (see Darwin's expression for t). Conservation of energy requires that $|t/t_o| \leq 1$. This condition is satisfied if the unscattered forward wave has an amplitude $t_o(1 - \sigma^2)^{1/2}$ and if the scattered forward wave has an amplitude $t_o(-i\sigma)$ assuming that $|\sigma| \leq 1$. The single-scattering coefficient t is therefore given by

$$t = (1 - s^2)^{1/2}[(1 - \sigma^2)^{1/2} - i\sigma] \tag{3.13}$$

By comparison of Equations 3.10 and 3.12 with 3.13, it is noted that $\exp\ (i\ u) = (1 - \sigma^2)^{1/2} - i\sigma$. Since the scattered portion of the forward wave is identical for normal and parallel polarizations (see Equation 3.6), $\sigma_n = \sigma_P = \sigma$. For scattering by a single plane array in free space, $u_n = u_p = u$. However, it will be shown later that, for scattering by a lattice plane in a phase-ordered medium, $u_n \neq u_p$. In order not to confuse the two cases, let $(1 - \sigma^2)^{1/2} - i\sigma = \exp\ (i\ u_n)$ where $\sigma_n = \sigma_, = \sigma$. However, $(1 - \sigma^2)^{1/2} - i\sigma = \cos(\arcsin \sigma) - i \sin(\arcsin \sigma) = \exp\ (-i \arcsin \sigma) = \exp\ (i\ u_n)$.

Accordingly,

$$\sigma = \sigma_n = \sigma_p = - \sin u_n \tag{3.14}$$

An explicit expression for u_n, in terms of the scattering from each electron, is determined by noting in Equation 3.14 that $u_n \approx -\sigma$ for $|\sigma| << 1$. Therefore, from Equation 3.8, a possible solution for u_n is given by

$$u_n - u_p = u = - G_n\ F(\mathbf{0}, h)\ M_o\lambda/\sin\psi \tag{3.15}$$

The parameter σ is therefore given by

$$\sigma = \sigma_n = \sigma_p = - \sin u_n = \sin[G_n\ F(\mathbf{0}, h)\ M_o\lambda/\sin\psi] \tag{3.16}$$

From Equation 3.9 and by comparison of Equation 3.16 with Equation 3.8, a possible solution for s is given by

$$s = \begin{cases} s_n = -\sin\left[u_n \dfrac{F_T(\psi, h)}{F(0, h)} (\text{sign} \cos 2\psi)\right] \\ s_p = -\sin\left[u_n \dfrac{F_T(\psi, h)}{F(0, h)} \cos 2\psi\right] \end{cases} \tag{3.17}$$

where u_n is given by Equation 3.15. With reference to Equations 3.10 and 3.12, the single-scattering coefficients a and t for a single plane array in free space are given by

$$a = \begin{cases} a_n = i \sin\ [u_n(\text{sign} \cos 2\psi)\ (1/Z_A) F_T(\psi, h)] \cdot \exp\ (iu_n) \\ a_p = i \sin\ [u_n\ (\cos 2\psi)\ (1/Z_A) F_T(\psi, h)] \cdot \exp\ (iu_p) \end{cases} \tag{3.18}$$

$$t = \begin{cases} t_n = \cos\ [u_n(\text{sign} \cos 2\psi)\ (1/Z_A) F_T(\psi, h)] \cdot \exp\ (iu_n) \\ t_p = \cos\ [u_n(\cos 2\psi)\ (1/Z_A) F_T(\psi, h)] \cdot \exp\ (iu_p) \end{cases} \tag{3.19}$$

where $u = u_n = u_P$ is given by Equation 3.15. Equations 3.18 and 3.19 are significant improvements over Darwin's expressions. They are applicable for any angle of incidence at any wavelength of the electromagnetic spectrum (where the recoil of the electron can be neglected). Equations 3.18 and 3.19 exhibit the periodic properties which generally characterize the interference of waves.

Scattering by Bound Electrons of a Lattice Plane in a Phase-Ordered Medium

The single-scattering coefficients a and t for a lattice plane, in a phase-ordered medium consisting of parallel lattice planes of separation d (see FIG. 2), are not the same as for a single plane array in free space because the function G_n is modified when other lattice planes are brought in close proximity to the single array.

The modified function G_{no} for totally constructive interference may be calculated using Clausius-Mosotti, Lorentz-Lorenz, Onsager, or other mathematical models of the local polarization fields[18]. The modified function G_{no} for totally constructive interference is related to the permittivity ϵ_j of a continuous medium j by[15]

$$\epsilon_j/\epsilon_o = 1 + P_j/\epsilon_o T_1 = 1 + G_{no}(4\pi c^2/\omega^2)N = 1 + (4\pi c^2/\omega^2)\sum_s N_s (G_{no})_s \quad (3.20)$$

where

P_j = electronic or atomic polarization of medium j

N = number of scattering electrons per unit volume

N_s = number of scattering electrons per unit volume of the oscillator type s

ϵ_o = permittivity of free space

However, the modified function G_{no} is still not satisfactory. The modified function G_{no} does not consider the interference of the scattered waves since the interference is generally not totally constructive. Therefore, the single-scattering coefficients a and t must be modified by more than simply substituting G_{no} for G_n in Equations 3.18 and 3.19. Therefore the permittivity ϵ_j given by Equation 3.20 should not be confused with the effective permittivity ϵ'_j defined by Equation 2.23. Whereas the permittivity ϵ_j is the permittivity for a continuous medium j, ϵ'_j considers the actual interference of the scattered waves. In this context, the permittivity ϵ_j will be used to characterize the polarization properties of a continuous medium whereas ϵ'_j will refer to the effective permittivity of a phase-ordered medium which is not necessarily continuous. It will be assumed that the relative permeability $\mu_j/\mu_i = 1$ so that $\epsilon_j/\epsilon_i = (v_i/v_j)^2 = v_{ij}^2$ and $\epsilon'_j/\epsilon_i = (v'_{ij})^2$. The determination of the single-scattering coefficients a and t for phase-ordered media now follows.

Consider a wave, of wavelength λ (radian frequency $\omega = 2\pi v_i/\lambda$) which is incident at a grazing angle ψ at time $t = 0$ at the origin of the first lattice plane of medium j (see FIG. 10). If the interference of the waves, specularly transmitted and reflected by each lattice plane, were assumed to be totally constructive as in a continuous medium, then the wave would travel in time t the distance $\rho = v_j t$. The transmitted wave, normalized to unit amplitude, would therefore be given by $\exp[(i2\pi/\lambda)(v_i t - v_i\rho/v_j)]$. If there were no scattering at all, then the wave would propagate in the forward direction and be given by $\exp[(i\,2\pi/\lambda)(v_i t - \rho)]$. If the interference of the specularly scattered waves from each lattice plane is neglected but not necessarily assumed to be totally constructive, then the transmitted wave would be given by

$$(t/t_o)^{r-1}\exp[(i\,2\pi/\lambda)(v_i t - \rho)] = \exp[(i\,u\,(r-1)]\exp[(i\,2\pi/\lambda)(v_i t - \rho)]$$

where $t/t_o = \exp(i\,u)$ is defined by Equations 3.10 – 3.14 and $r-1$ is the number of lattice planes which are penetrated in travelling the distance ρ. If it is assumed that r can be propitiously chosen, so that the totally constructive wave can be equated with the wave for which interference is neglected, then

$$\exp[i\,u\,(r-1)]\exp[(i\,2\pi/\lambda)(v_i t - \rho)] = \exp[(i\,2\pi/\lambda)(v_i t - v_i\rho/v_j)] \quad (3.21)$$

Solving for u,

$$u = [2\pi\rho(1 - v_i/v_j)]/[\lambda(r-1)] \quad (3.22)$$

At x-ray wavelengths, the transmitted wave travels almost in the forward direction since the index of refraction is approximately unity and the scattering parameters σ and s are very small ($\approx 10^{-6}$). Darwin therefore assumed[12] that $r-1 = (\rho\sin\psi)/d$. However, this assumption is valid only if there were no scattering at all. Instead, the author has found that $r-1$ is given by

$$r-1 = \begin{cases} r_n - 1 = [(\rho\sin\psi)/d][2v_j/(v_i+v_j)], \\ \text{electric field polarized normal to the plane of incidence} \\ r_p - 1 = [(\rho\sin\psi)/d][2v_j/(v_i+v_j)][2/(2+\cos 2\psi_j - \cos 2\psi)] \\ \text{electric field polarized parallel to the plane of incidence} \end{cases} \quad (3.23)$$

where $\psi_j = \cos^{-1}(v_{ij}^{-1}\cos\psi)$ = angle of refraction for continuous media.

Equations 3.23 may be deduced by the following considerations. If the interference of the scattered waves were totally constructive, then the transmitted wave would propagate in the direction ψ_j defined above. If the actual interference were considered, then the transmitted wave would propagate in the direction ψ_j' given by Equation 2.22. Since the wave on the left-hand side of Equation 3.21 is constructed by neither the actual interference nor the totally constructive interference of scattered waves, its direction $(\psi+\Delta\psi)$ in travelling a path length ρ is not known. However, it is known that the wave's initial velocity in medium j is v_i and that its final velocity is v_j since it is equated with a wave travelling with velocity v_j. Therefore the wave's final velocity v_j relative to its average velocity is $2v_j/(v_i+v_j)$. Let the wave's direction of propagation $(\psi+\Delta\psi_n)$, determined solely by its change in velocity and its initial direction ψ, be specified by the requirement that $\sin(\psi+\Delta\psi_n) \rightarrow \sin\psi$ as $v_i \rightarrow (v_i+v_j)/2 \rightarrow v_j$. Accordingly,

$$\sin(\psi + \Delta\psi_n) = 2v_j\sin\psi/(v_i + v_j) \quad (3.24)$$

Setting $r_n - 1 = \rho\sin(\psi+\Delta\psi_n)/d$, Equation 3.23a is obtained. From Equation 3.6 it is seen that the wave scattered by each electron in the forward direction is identical for normal and parallel polarizations since $\cos\phi = 1$. For a single plane array, the phase of the total transmitted wave is independent of polarization (see Equation 3.19) because the transmitted wave is in the forward direction. Scattering of the transmitted wave by several lattice planes causes a change in its velocity with a resultant change in its direction (see Equation 3.24). Since the transmitted wave is no longer in the forward direction and $\cos\phi \neq 1$, an additional change in direction occurs for parallel polarization of the wave. By analogy with Equation 3.24, let the direction of propagation $(\psi + \Delta\psi_p)$ for parallel polarization, determined by its initial direction ψ, its change in velocity, and its deviation from unity of $\cos\phi$, be given by

$$\sin(\psi + \Delta\psi_p) = [\sin\psi][2v_j/(v_i+v_j)]\{2(\cos\phi)_j/[(\cos\phi)_i + (\cos\phi)_j]\} \quad (3.25)$$

where $(\cos\phi)_i$ and $(\cos\phi)_j$ are the initial and final values of $\cos\phi$. The wave's initial direction is ψ and its final direction is ψ_j since it is equated to a wave for which interference is totally constructive. The final value $(\cos\psi)_j = 1$ since this is characteristic of a transmitted wave with totally constructive interference. The initial value $(\cos\phi)_i$ must have the property that $(\cos\phi)_i \rightarrow 1$ as $\psi_j \rightarrow \psi$. Since $\phi = 2\psi$ (see Equation 3.9), $(\cos\phi)_i$ should also be a function of $\cos 2\psi_j$ and $\cos 2\psi$.

A function which satisfies these two requirements is given by $(\cos\phi)_i = 1 + \cos 2\psi_j - \cos 2\psi$. Substituting $(\cos\phi)_j = 1$ and $(\cos\phi)_i = 1 + \cos 2\psi_j - \cos 2\psi$ into Equation 3.25 and setting $r_p - 1 = \rho \sin(\psi \text{ ' } \Delta\psi_p)/d$, Equation 3.23*b* is obtained.

Equations 3.23 can also be derived by inductive arguments independent of the above deductive derivation. For example, it will be shown in Section 4.0 that Equation 2.14 satisfys conservation of energy requirements and reduces to Fresnel's reflection coefficients, for normal and parallel polarizations, as the spacing d between lattice planes approaches zero, provided that the values of *r* given by Equation 3.23 are used in deriving *a* and *t*.

The single-scattering coefficients *a* and *t* for a lattice plane in phase-ordered media are determined by solving for *u* and substituting *u* into Equations 3.16 – 3.19. Substituting Equations 3.23 into Equation 3.22, *u* is given by

$$u=\begin{cases} u_n=2\pi d(1-v_{ij}^2)/(2\lambda\sin\psi)=\bar{u} \\ u_p=2\pi d(1-v_{ij}^2)(\sin^2\psi+v_{ij}^2\cos^2\psi)/(2\lambda\sin\psi) \\ \qquad =\bar{u}(\sin^2\psi+v_{ij}^{-2}\cos^2\psi) \end{cases} \tag{3.26}$$

where

$\bar{u}$ = phase difference defined by Equation 2.3 and derived independently in Section 2.0.

$v_{ij}^2 = (v_i/v_j)^2 = \epsilon_j/\epsilon_i$

$\sin^2\psi + v_{ij}^{-2}\cos^2\psi = 1 + (1/2)(\cos 2\psi_j - \cos 2\psi)$

Substituting Equation 3.26*a* into Equation 3.14, σ is given by

$$\sigma = \sigma_n = \sigma_p = -\sin u_n = -\sin \bar{u} \tag{3.27}$$

Substituting Equation 3.26 into an equation analogous to that of Equation 3.17,

$$s=\begin{cases} s_n=-\sin[u_n(1/Z_A)F_T(\psi,h)]=-\sin[u(1/Z_A)F_T(\psi,h)] \\ s_p=-\sin[u_n(-1)\overline{(\cos 2\psi)}(1/Z_A)F_T(\psi,h)]=-\sin[u(\sin^2\psi-v_{ij}^{-2}\cos^2\psi)(1/Z_A)F_T(\psi,h)] \end{cases} \tag{3.28}$$

where

$\cos\bar{\phi} = \overline{\cos 2\psi} = (\cos 2\psi_j + \cos 2\psi)/2 = -\sin^2\psi + v_{ij}^{-2}\cos^2\psi$

sign *s* has been chosen so that $s_n = s_p$ for normal incidence and so that the reflected electric field intensity is positive in the same direction as the incident electric field intensity.

Substituting Equation 3.28 into Equations 3.10 and 3.12, the single-scattering coefficients *a* and *t* for a lattice plane in phase-ordered media are given by

$$a=\begin{cases} a_n=i\sin[u_n(1/Z_A)F_T(\psi,h)]\cdot\exp(iu_n) \\ \quad =i\sin[\bar{u}(1/Z_A)F_T(\psi,h)]\cdot\exp(i\bar{u}) \\ a_p=i\sin[u_n(-1)\overline{(\cos 2\psi)}(1/Z_A)F_T(\psi,h)]\cdot\exp(iu_p) \\ \quad =i\sin[\bar{u}(\sin^2\psi-v_{ij}^{-2}\cos^2\psi)(1/Z_A)F_T(\psi,h)]\cdot\exp[i\bar{u}(\sin^2\psi+v_{ij}^{-2}\cos^2\psi)] \end{cases} \tag{3.29}$$

$$t=\begin{cases} t_n=\cos[u_n(1/Z_A)F_T(\psi,h)]\cdot\exp(iu_n) \\ \quad =\cos[\bar{u}(1/Z_A)F_T(\psi,h)]\cdot\exp(i\bar{u}) \\ t_p=\cos[u_n(-1)\overline{(\cos 2\psi)}(1/Z_A)F_T(\psi,h)]\cdot\exp(iu_p) \\ \quad =\cos[\bar{u}(\sin^2\psi-v_{ij}^{-2}\cos^2\psi)(1/Z_A)F_T(\psi,h)]\cdot\exp[i\bar{u}(\sin^2\psi+v_{ij}^{-2}\cos^2\psi)] \end{cases} \tag{3.30}$$

Equations 3.29 and 3.30 should be compared with Darwin's single-scattering coefficients at x-ray wavelengths, namely[12]

$$\begin{matrix} a_n=a_p/\cos 2\psi=i\bar{u}(1/Z_A)F_T(\psi,h) \\ t_n=t_p=1+i\bar{u} \end{matrix}, \quad v_{ij}\approx 1 \tag{3.31}$$

At x-ray wavelengths, $v_{ij} \approx 1$ for all media. Equations 3.29 and 3.30 reduce to Equations 3.31 if terms of higher order than $(1-v_{ij})$ are ignored. Equations 3.29 and 3.30 therefore contain higher order corrections to the present state of the art of x-ray spectroscopy.

However, of far greater significance, Equations 3.29 and 3.30 are valid for any angle of incidence and for any wavelength of the electromagnetic spectrum (where the recoil of the electron can be neglected). Equations 3.29 and 3.30 exhibit periodic properties previously unreported for scattering of electromagnetic waves in phase-ordered media. From Equations 3.29 and 3.30 for normal polarization, it is noted that total reflection occurs for $\bar{u}(1/Z_A)F_T(\psi,h) = \pm(2M+1)\pi/2$, $M = 0, 1, 2, \ldots$. Total transmission occurs for $\bar{u}(1/Z_A)F_T(\psi,h) + \pm M\pi$, $M = 0, 1, 2, \ldots$. With the exception of the factor $(1/Z_A)F_T(\psi,h)$, which accounts for the interference of waves scattered by the electrons comprising a single scattering center (atom, molecule, or unit cell of a crystal), these conditions agree with those given by Equations 2.5 and 2.6 for total reflection and total transmission respectively. Neither Equations 3.29 and 3.30 nor Equations 2.5 and 2.6 consider the multiple scattering of the waves scattered by every lattice plane of the medium. A more complete treatment of the properties of reflection and transmission in phase-ordered media is given in the following section where multiple scattering is considered.

4.0 REFLECTION AND TRANSMISSION IN SEMI-UNBOUNDED MEDIA

The complete quantitative description of steady-state electromagnetic waves in phase-ordered, semi-unbounded media is determined by substitution of the single-scattering coefficients *a* and *t*, given by Equations 3.29 and 3.30, into Equations 2.10 – 2.14. The secondary refractive parameters are determined by substitution of Equations 3.29 and 3.30 into Equations 2.15 – 2.23.

It is convenient to define the parameter U by

$$U=\begin{cases} U_n=u_n(1/Z_A)F_T(\psi,h) \\ U_p=u_n(\sin^2\psi-v_{ij}^2\cos^2\psi)(1/Z_A)F_T(\psi,h) \end{cases} \tag{4.1}$$

where u_n is given by Equation 3.26*a*. The quantity *A* of Equation 2.10 reduces to

$$A=\begin{cases} A_n=\cos(V-u_n)/\cos U_n \\ A_p=\cos(V-u_p)/\cos U_p \end{cases} \tag{4.2}$$

where *V*, *u*, and *U* are given by Equations 2.1, 3.26, and 4.1 respectively. The multiple-scattering amplitude delay coefficient *X*, defined by Equation 2.11 reduces to

$$X=\begin{cases} X_n=\dfrac{\cos(V-u_n)}{\cos U_n}\pm\left[\dfrac{\cos^2(V-u_n)}{\cos^2 U_n}-1\right]^{1/2}, |X_n|\le 1 \\ X_p=\dfrac{\cos(V-u_p)}{\cos U_p}\pm\left[\dfrac{\cos^2(V-u_p)}{\cos^2 U_p}-1\right]^{1/2}, |X_p|\le 1 \end{cases} \tag{4.3}$$

where V, u, and U are given by Equations 2.1, 3.26, and 4.1 respectively. The multiple-scattering reflection coefficient S_1/T_1, given by Equation 2.14, reduces to

$$S_1/T_1=\begin{cases}(S_1/T_1)_n=\dfrac{i\sin U_n\cdot\exp(iu_n)}{1-X_n\cos U_n\cdot\exp[-i(V-u_n)]}\\[2ex](S_1/T_1)_p=\dfrac{i\sin U_p\cdot\exp(iu_p)}{1-X_p\cos U_p\cdot\exp[-i(V-u_p)]}\end{cases}\qquad(4.4)$$

In Equation 4.4, the reflected electric field intensity S_1 is positive for the same direction as the incident electric field intensity T_1. Equations 4.3 and 4.4 determine the properties of steady-state electromagnetic waves in phase-ordered, semi-unbounded discrete media subject to the following conditions:

1. The recoil of the electrons is negligible.
2. The scattering cross-section of each electron is sufficiently small so that the mutual interactions between neighboring atoms in any lattice plane are negligible.
3. Pertubations or changes in the quantum sates of the medium are negligible.
4. The relative permeability $\mu_j/\mu_i = 1$.

The coefficients S_1/T_1 and X are functions of the physical parameters d/λ, ψ, ν_U, and $(1/Z_A)F_T(\psi,h)$. A complete quantitative description of electromagnetic waves in phase-ordered, semi-unbounded media is obtained by evaluating Equations 4.3 and 4.4 for all possible combinations and values of the physical parameters. In this section, each of the following cases will be considered separately:

1. ν_U is real
2. $V >> 1, |\bar{u}|<<1$
3. $V << 1, 0 \leq |\bar{u}| < \infty$
4. $0 \leq V < \infty, |\bar{u}| << 1$
5. $0 \leq V <\infty, 0 \leq |\bar{u}| <\infty$.

Case 1 includes the general conditions for total reflection and total transmission in dielectrics with zero absorption loss. Case 2 includes the special case of continuous media and is applicable to most electromagnetic wave-interactions. Case 3 is applicable to very low grazing angles of incidence and to the propagation in metals. Case 4 includes the case of Bragg reflection at x-ray wavelengths. Case 5 considers cases not included by Cases 2–4.

ν_U is real

Consider a wavelength λ sufficiently far removed from an absorption band so that $\nu_U^2 = \epsilon_j/\epsilon_i$ is a real quantity. Media, characterized by Im $\nu_U = 0$, shall be referred to as "dielectrics." The reflection coefficient S_1/T_1 and the amplitude delay coefficient X are now determined for dielectrics.

First, it will be shown that

$$|S_1/T_1|^2 = 1,\ \cos^2(V-u)/\cos^2U \geq 1,\ \nu_U \text{ real}\qquad(4.5)$$

where it is understood that V, u, and U are given by Equations 2.1, 3.26, and 4.1 respectively. For the conditions of Equation 4.5, Equation 4.3 reduces to

$$X \text{ real},\ -1 \leq X \leq 1,\ \cos^2(V-u)/\cos^2U \geq 1,\ \nu_U \text{ real}\qquad(4.6)$$

Substituting the conditions of Equation 4.6 into Equation 4.4, $|S_1/T_1|^2 = \sin^2U/|1-\exp[-i(V-u)]\cdot(\cos U)\cdot X|^2 = \sin^2U/\sin^2U = 1$. The most general conditions for total reflection in dielectrics are therefore given by Equation 4.5. Equation 4.5 includes the cases of total internal reflection at the angle ψ_c (see discussion for $V << 1, |\bar{u}| << 1$), Bragg reflection at x-ray wavelengths (see discussion for $V \approx N\pi$, $|\bar{u}| << 1$), and the case qualitatively given by Equation 2.5 for low grazing angles of incidence (see discussion for $V << 1, 0 \leq |\bar{u}| <\infty$).

It follows from Equation 4.5 that

$$|S_1/T_1|^2 < 1,\ \cos^2(V-u)/\cos^2U < 1,\ \nu_U \text{ real}\qquad(4.7)$$

For the conditions of Equation 4.7, Equation 4.3 reduces to

$$X \text{ complex},\ |X| = 1,\ \cos^2(V-u)/\cos^2U < 1,\ \nu_U \text{ real}\qquad(4.8)$$

For the conditions of Equation 4.8, the values of X are confined to the perimeter of the unit circle in the complex X plane. Substituting Equation 4.3 into Equation 4.4, the intensity of the reflection coefficient, with or without absorption loss, is given by

$$|S_1/T_1|^2=\left|\frac{\sin U}{\sin(V-u)\mp[\cos^2U-\cos^2(V-u)]^{1/2}}\right|^2\qquad(4.9a)$$

where it is understood that the $\mp$ roots of Equation **4.9*a*** correspond to the $\pm$ roots of Equation 4.3, the choice of sign being determined by the requirement $|S_1/T_1|^2 \leq 1$. For ν_U real, Equation 4.9*a* reduces to

$$|S_1/T_1|^2 = (\sin U/\sin(V-u) \mp [\cos^2U - \cos^2(V-u)]^{1/2})^2,\ \nu_U \text{ real}\qquad(4.9b)$$

With reference to Equation 4.9*b*,

$$\lim_{\sin U\to 0}|S_1'/T_1|^2=\left[\frac{\sin U}{2\sin(V-u)}\right]^2=0\qquad(4.10)$$

The necessary and sufficient condition for zero reflection (total transmission) in phase-ordered dielectrics is therefore given by

$$U = U_{0,M} = M\pi\quad M = 0, \pm 1, \pm 2, \ldots,\ \nu_U \text{ real},\ S_1/T_1 = 0\qquad(4.11)$$

where U is given by Equation 4.1. For zero reflection, Equation 4.3 reduces to

$$X = \exp[\pm i(V-u)],\ S_1/T_1 = 0\qquad(4.12)$$

Equation 4.11 includes the cases $\nu_U = 1$, Brewster's angle for parallel polarization (see discussion for $V << 1, |\bar{u}| << 1$), and the case qualitatively given by Equation 2.6 for low grazing angles of incidence (see discussion for $V << 1, 0 \leq |\bar{u}|<\infty$).

The allowable values of the transmission X in phase-ordered dielectrics are plotted in the complex X plane in FIG. 8*a*. The values of X is specified by the vector drawn from the origin O to a point on the plot. The vector X for phase-ordered media generally traverses an infinite number of cycles about the path shown in FIG. 8*b* as the grazing angle of incidence is varied from 0 to $\pi/2$.

$V << 1, |\bar{u}| << 1$

Most electromagnetic wave-interactions, excluding near-zero grazing angles of incidence, Bragg reflection at x-ray wavelengths, and propagation in metals, are characterized by the joint phase conditions

$$V << 1, |\bar{u}| << 1\qquad(4.13)$$

where V and u are defined by Equations 2.1 and 2.3 respectively. These conditions include the special case of totally constructive interference in continuous media defined by

$$V = \bar{u} = 0 \qquad (4.14)$$

Present electromagnetic theory [2-4], with the exception of theories describing Bragg reflection at x-ray wavelengths[1, 12], is based on Conditions 4.14. In this sub-section, the relatively small corrections introduced by Conditions 4.13 are considered.

For $V << 1$, the normalized average intensity of the atomic structure factor reduces to

$$F_T^2(\psi,h)/F^2(0,h) = (1/Z_A)^2 F^2(\psi,h)\exp[-2M_T(\psi,h)] \approx (1 - h^2R_o^2/3)(1-2M_T)$$

$$\approx 1 - (V^2/3d^2)(2R_o+6\delta_{rms})^2 = 1 - (V^2/3)(2R_{oT}/d)^2,\ V << 1 \qquad (4.15)$$

where

R_o = electronic radius of gyration[19] of a particle (atom, molecule, or unit cell) at rest about its electronic center of mass = root mean squared orbital radius of the electrons comprising the particle at rest. If d is the spacing between particles, then $2R_o/d \leqslant 1$. If d is the spacing between particles in amorphous media (see Equation 2.7), then $R_o \approx (d/2)(2/5)^{1/2}$ assuming as a first approximation that the electron density is uniform in a sphere of radius $d/2$. For continuous media, $2R_o/d = 1$.

$M_T(\psi,h) = 8\pi^2\,\overline{\delta_T^2}(\sin^2\psi)/\lambda^2 = 2\,V^2\,\overline{\delta_T^2}/d^2$ (see Equation 3.5)

$\delta_{rms} = (\overline{\delta_T^2})^{1/2}$ = root mean square displacement of the particle in thermal agitation above and below its average position in the atomic plane under consideration.

$R_{oT} = R_o(1 + 3\,\delta_{rms}/R_o)$ = electronic radius of gyration of a particle in thermal agitation. For a solid at room temperature, $R_{oT} \approx R_o$. For a gas, $R_{oT} = R_o + 3d$. (see discussion following Equation 2.7).

For purposes of simplifying the equations which are to follow for Conditions 4.13, it is convenient to define the following parameters:

$$A = \begin{cases} A_n = 1 + (1/2)V^2(Q-1) \\ \quad + (1/24)V^4[(Q-1)(Q-1+Q^2) - Q^2(2R_{oT}/d)^2] \\ A_p = 1 + (1/2)V^2(Q-1) \\ \quad + (1/24)V^4[(Q-1)(Q-1+Q_-^2) - Q_-^2(2R_{oT}/d)^2] \end{cases} \qquad (4.18)$$

where terms containing factors of higher order than $(2\pi d/\lambda)^4$ have been neglected. For Conditions 4.13, it is noted from Equation 4.18 that $|A-1| << 1$. However, for $|A-1| << 1$, the transmission X of Equations 2.11 reduce to

$$X = A - (A^2-1)^{1/2} = 1 + (1/2)\,2\,(A-1) - \{2(A-1)[1 + (A-1)/2]\}^{1/2}$$

$$\approx 1 - [2(A-1)]^{1/2} + (1/2)\,2(A-1) - (1/8)2(A-1)[2(A-1)]$$

$$\approx \exp - [2(A-1)]^{1/2} + (1/24)2(A-1)[2(A-1)]^{1/2}$$

$$\approx \exp\{-[2(A-1)]^{1/2}[1-(1/24)2(A-1)]\}\ |A-1| << 1 \qquad (4.19)$$

Substituting Equation 4.18 into Equation 4.19,

$$X = \begin{cases} X_n = \exp[-V\{(Q-1)[1+(V^2/12)(Q-1+Q^2)] \\ \quad -(V^2/12)Q^2(2R_{oT}/d)^2\}^{1/2}\{1-(V^2/24)(Q-1)\}] \\ X_p = \exp[-V\{(Q-1)[1+(V^2/12)(Q-1+Q_-^2)] \\ \quad -(V^2/12)Q_-^2(2R_{oT}/d)^2\}^{1/2}\{1-(V^2/24)(Q-1)\}] \end{cases}$$

$$X = \begin{cases} X_n \approx \exp\{-V[(Q-1)(1+V^2Q^2/12) \\ \quad -(V^2/12)Q^2(2R_{oT}/d)^2]^{1/2}\} \\ X_p \approx \exp\{-V[(Q-1)(1+V^2Q_-^2/12) \\ \quad -(V^2/12)Q_-^2(2R_{oT}/d)^2]^{1/2}\} \end{cases},\ V<<1,\ |\bar{u}|<<1 \qquad (4.20)$$

where terms containing factors of higher order than $(2\pi d/\lambda)^4$ have been neglected. With reference to FIG. 8*a* and Equation 4.20, the only allowable values of X, for Conditions 4.13 and Im $v_{ij} = 0$, are those which are on the real axis or on the unit circle in the vicinity of $X=1$. For Conditions 4.13 and Im $v_{ij} \neq 0$, the allowable values of X in the complex X plane are within the unit circle in the vicinity of $X=1$.

Substituting Equation 4.20 into Equation 2.16, the effective propagation constant γ_{jz}' in the direction of the Z axis in the medium j is given by

$$\gamma_{jz}' = \begin{cases} (\gamma_{jz}')_n = (V/d)[(Q-1)(1+V^2Q^2/12) - (V^2/12)Q^2(2R_{oT}/d)^2]^{1/2} \\ (\gamma_{jz}')_p = (V/d)[(Q-1)(1+V^2Q_-^2/12) - (V^2/12)Q_-^2(2R_{oT}/d)^2]^{1/2} \end{cases},\ V<<1,\ |\bar{u}|<<1 \qquad (4.21)$$

$$Q = 2\,\bar{u}/V = 2\,u_n/V = (1-v_{ij}^2)/\sin^2\psi$$

$$Q_+ = (2\,\bar{u}/V)(\sin^2\psi + v_{ij}^{-2}\cos^2\psi) = (2\,u_n/V)(u_p/u_n) = (1-v_{ij}^2)(1+v_{ij}^{-2}\cot^2\psi) \qquad (4.16)$$

$$Q_- = (2\bar{u}/V)(\sin^2\psi - v_{ij}^{-2}\cos^2\psi) = (2u_n/V)(U_p/U_n) = (1-v_{ij}^2)(1-v_{ij}^{-2}\cot^2\psi)$$

where u and U are given by Equations 3.26 and 4.1 respectively. Equations 4.16 satisfy the identity

$$(1 - Q_+/2)^2 = 1 - Q + Q_+^2/4 \qquad (4.17)$$

For Conditions 4.13 and with the use of Equations 4.15–4.17, the parameter A of Equation 4.2 reduces to

$$A = (1 - (1/2)(V-u)^2 + (1/24)(V-u)^4 - \ldots./1 - (1/2)U^2 + (1/24)U^4 - \ldots..)$$

$$\approx 1 - (1/2)(V-u)^2 + (1/24)(V-u)^4\ 1 + (1/2)U^2 + (5/24)U^4$$

$$\approx 1 - (1/2)(V-u)^2 + (1/2)U^2 + (1/24)(V-u)^4 + (5/24)U^4 - (1/4)U^2(V-u)^2 + \ldots$$

By definition of γ_{jz}', it should follow that

$$\lim_{d\to 0} \gamma_{jz}' = \gamma_{jz} =$$

propagation constant in the direction of the z axis in a continuous medium j.

From Equation 4.21,

$$\lim_{d\to 0} \gamma_{jz}' = (V/d)(Q-1)^{1/2} = (2\pi/\lambda)\sin\psi[(1/\sin^2\psi)(1-v_{ij}^2)-1]^{1/2} = (i2\pi/\lambda)\sin\psi_j = \gamma_{jz} \qquad (4.22)$$

where

$\sin\psi_j = (1 - v_{ij}^{-2}\cos^2\psi)^{1/2}$ = Snell's law for the angle ψ_j of refraction in continuous media (see FIG. 2*b*).

To the best of the author's knowledge, Equation 4.22 marks the first time that Snell's law of refraction has been formally derived from the interference of waves scattered by each electron. Substituting Equation 4.22 into Equation 4.21,

$$\gamma_{jz}'=\begin{cases}(\gamma_{jz}')_n=\gamma_{jz}\{1+(V^2Q^2/12)[1-(2R_{oT}/d)^2/(Q-1)]\}^{1/2}\\(\gamma_{jz}')_p=\gamma_{jz}\{1+(V^2Q_{-}^2/12)[1-(2R_{oT}/d)^2/(Q-1)]\}^{1/2}\end{cases}\quad(4.23)$$

The effective secondary parameters γ_j', v_{ij}', $\cos\psi_j'$, and ϵ_j' are determined by substitution of Equation 4.20 into Equations 2.21–2.23. Accordingly,

$$\gamma_j'=\begin{cases}(\gamma_j')_n=\gamma_j\{1-[(V^2Q^2\sin^2\psi)/12v_{ij}^2][Q-1-(2R_{oT}/d)^2]\}^{1/2}\\(\gamma_j')_p=\gamma_j\{1-[(V^2Q_{-}^2\sin^2\psi)/12v_{ij}^2][Q-1-(2R_{oT}/d)^2]\}^{1/2}\end{cases},\ V<<1,\ |\bar{u}|<<1\quad(4.24)$$

where $\gamma_j=(i\,2\pi/\lambda)v_{ij}$ = propagation constant in a continuous medium j.

$$v_{ij}'/v_{ij}=(\cos\psi_j'/\cos\psi_j)^{-1}=(\epsilon_j'/\epsilon_j)^{1/2}=\gamma_j'/\gamma_j\quad(4.25)$$

The secondary parameters differ from those for continuous media by the percentage deviation given by

$$(\gamma_j'-\gamma_j)/\gamma_j=(v_{ij}'-v_{ij})/v_{ij}=(\cos\psi_j'-\cos\psi_j)/\cos\psi_j$$
$$=\begin{cases}[(\gamma_j')_n-\gamma_j]/\gamma_j=[(V^2Q^2\sin^2\psi)/24v_{ij}^2][Q-1-(2R_{oT}/d)^2]\\[(\gamma_j')_p-\gamma_j]/\gamma_j=[(V^2Q_{-}^2\sin^2\psi)/24v_{ij}^2][Q-1-(2R_{oT}/d)^2]\end{cases};\ V<<1,\ |\bar{u}|<<1\quad(4.26)$$

The minimum percentage deviation occurs at normal incidence ($\psi=\pi/2$) for $v_{ij}>1$ and at an angle $\psi=\sin^{-1}[-1-v_{ij}^2/1+(2R_{oT}/d)^2]$ for $v_{ij}<1$. At grazing angles of incidence characterized by $\sin^2\psi<|(1-v_{ij}^2)/\ 1+(2R_{oT}/d)^2|$, the percentage deviation increases with decreasing grazing angles of incidence and is given by Equation 4.26 until the condition $|\bar{u}|<<1$ is no longer satisfied. For visible radiation, of wavelength $\lambda=5\times10^{-7}$m in free space, which is incident on fused quartz [$d=4.9\times10^{-10}m$, $v_{ij}=1.5$, $(2R_{oT}/d)^2\approx2/5$] at normal incidence, the percentage deviation$|[(\gamma_j'-\gamma_j)/\gamma_j]|=2.9\times10^{-4}$%. If the lattice spacing d is increased by a factor of 20 (as might occur in a dilute liquid solution), then for the same conditions and dielectric properties as the previous example, the percentage deviation at normal incidence is 0.1 percent. To the best of the author's knowledge, Equation 4.26 marks the first time that the error, caused by the substitution of Conditions 4.14 for Conditions 4.13 has been calculated. It should be noted that every phase-ordered medium exhibits frequency dispersion even if the relative phase velocity v_{ij} is a real constant, unless the medium is continuous.

Discrete, phase-ordered media are birefringent (double-refracting) because the normal and parallel components of polarization have different velocities and different directions of propagation as shown by Equation 4.26. For example, consider visible radiation of wavelength $\lambda=5\times10^{-7}$m which is incident at a grazing angle $\psi=\pi/4$ radians on fused quartz of the previous example. Fused quartz is generally considered not to be birefringent. However, according to the above theory, the incident beam should separate into two orthogonally polarized beams whose difference in directions of propagation, $(\psi_j')_n-(\psi_j')_p=\Delta\psi$, is given by

$$\Delta\psi\approx|\cos(\psi_j')_n-\cos(\psi_j')_p|\ /\ \sin\psi_j=\cot\psi_j\,|(v_{ij}')_n-(v_{ij}')_p|/v_{ij},\ \Delta\psi<<1\quad(4.27)$$

Substituting into Equations 4.26 and 4.27 for the above example,

$[(v_{ij}')_n-(v_{ij}')_p]/v_{ij}=2.34\times10^{-4}$% and $\Delta\psi=4.43\times10^{-6}$ radians. This result should be compared to the birefringence of crystalline quartz whose anisotropy $[(v_{ij})_E-(v_{ij})_O]/v_{ij}=0.6$% where E and O refer to directions perpendicular and parallel to the optic axis[21]. For the same radiation incident on crystalline quartz at an angle $\pi/4$ radians with respect to the optic axis of the crystal, $\Delta\psi=6.8\times10^{-3}$ radians. The phase shift $\Delta\phi$ between the two orthogonal components of polarization per unit path length L is given by

$$\Delta\phi/L=(2\pi/\lambda)|(v_{ij}')_n-(v_{ij}')_p|\quad(4.28)$$

For the preceding examples for fused quartz and crystalline quartz, $\Delta\phi/L=0.37$ radians/cm and 1.1×10^3 radians/cm respectively. The birefringence of fused quartz is therefore small compared to crystalline quartz. Nevertheless, the birefringence of amorphous, phase-ordered materials may find application when anisotropic materials are not available.

The reflection coefficient S_1/T_1 for Conditions 4.13 may be determined directly, without the use of secondary parameters, by substituting Equation 4.20 into Equation 4.4 and expanding sinU, cosU, and $\exp[-i(V-u)]$ in a Taylor series expansion. Since the denominator of Equation 4.4 involves the difference of two quantities which are approximately unity for Conditions 4.13, it is important that terms of the order of $(2\pi d/\lambda)^3$ or larger be retained for accurate results. Substituting Equation 4.20 and the Taylor series expansions into Equation 4.4,

$$S_1/T_1=\begin{cases}(S_1/T_1)_n=\dfrac{i(U_n-U_n^3/6)[1+iu_n-(1/2)u_n^2-iu_n^3/6]}{1-(1-U_n^2/2)[1-i(V-u_n)-(1/2)(V-u_n)^2+(i/6)(V-u_n)^3]X_n}\\(S_1/T_1)_p=\dfrac{i(U_p-U_p^3/6)[1+iu_p-(1/2)u_p^2-iu_p^3/6]}{1-(1-U_p^2/2)[1-i(V-u_p)-(1/2)(V-u_p)^2+(i/6)(V-u_p)^3]X_p}\end{cases},\ V\ll1,|\bar{u}|\ll1\quad(4.29)$$

where X_n and X_p are the Taylor series expansions of Equation 4.20.

Definition of Equation 4.30

$$\lim_{d\to 0}\left(\frac{S}{T}\right)_n=\frac{i\mu_n}{iV(1-Q)^{1/2}+i(V-\mu_n)}=\frac{\frac{2}{V}(i\mu_n)}{\left(\frac{2}{V}\right)[iV(1Q)^{1/2}+i(V-\mu_n)]}=\frac{Q}{2(1-Q)^{1/2}+2-Q}$$

$$=\frac{Q}{[1+(1-Q)^{1/2}]^2}=\frac{Q[1-(1-Q)^{1/2}]}{[1+(1-Q)^{1/2}]^2[1-(1-Q)^{1/2}]}=\frac{Q[1-(1-Q)^{1/2}]}{Q[1+(1-Q)^{1/2}]}$$

$$=\frac{[1-(1-Q)^{1/2}]\sin\psi}{[1+(1-Q)^{1/2}]\sin\psi}=\frac{\sin\psi-(\sin^2\psi-1+v_{ij})^{1/2}}{\sin\psi+(\sin^2\psi-1+v_{ij})^{1/2}}=\frac{\sin\psi-v_{ij}\left(1-\frac{\cos^2\psi}{v_{ij}^2}\right)^{1/2}}{\sin\psi+v_{ij}\left(1-\frac{\cos^2\psi}{v_{ij}^2}\right)^{1/2}}$$

$$=\frac{\sin\psi-v_{ij}\sin\psi_j}{\sin\psi+v_{ij}\sin\psi_j}$$

$$\lim_{d\to 0}\left(\frac{S}{T}\right)_p=\frac{i\mu_n(\sin^2\psi-v_{ij}^2\cos^2\psi)}{iV(1-Q)^{1/2}+i(V-\mu_p)}=\frac{\frac{2}{iV}\mu_n\left(\sin^2\psi-\frac{\cos^2\psi}{\mu_{ij}^2}\right)}{\frac{2}{iV}[iV(1-Q)^{1/2}+i(V-\mu_p)]}$$

$$=\frac{Q\left(\sin^2\psi-\frac{\cos^2\psi}{v_{ij}^2}\right)}{2(1-Q)^{1/2}+2-Q\left(\sin^2\psi+\frac{\cos^2\psi}{v_{ij}^2}\right)}=\frac{\sin^2\psi\left[Q\left(\sin^2\psi\frac{\cos^2\psi}{v_{ij}^2}\right)\right]}{\sin^2\psi\left[2(1-Q)^{1/2}+2-Q\left(\sin^2\psi+\frac{\cos^2\psi}{v_{ij}^2}\right)\right]}$$

$$=\frac{\sin^2\psi-v_{ij}^2\sin^2\psi-\frac{\cos^2\psi}{v_{ij}^2}+\cos^2\psi}{2\sin\psi(\sin^2\psi-1+v_{ij}^2)^{1/2}+2\sin^2\psi-\sin^2\psi+v_{ij}^2\sin^2\psi-\frac{\cos^2\psi}{v_{ij}^2}+\cos^2\psi}$$

$$=\frac{\sin^2\psi_j-v_{ij}^2\sin^2\psi}{2v_{ij}\sin\psi\cdot\sin\psi_j+\sin^2\psi+\sin^2\psi_j+v_{ij}^2\sin^2\psi}=\frac{(\sin\psi_j-v_{ij}\sin\psi)(\sin\psi_j+v_{ij}\sin\psi)}{(\sin\psi_j+v_{ij}\sin\psi)^2}$$

$$=\frac{\sin\psi_j-v_{ij}\sin\psi}{\sin\psi_j+v_{ij}\sin\psi}$$

Equation 4.29 gives the reflection coefficient for phase-ordered, semi-unbounded media subject to Conditions 4.13. An essential property of Equation 4.29 is that it must reduce to Fresnel's formulae for Conditions 4.14. In other words, it is essential that

$$\lim_{d\to 0} S_1/T_1=S/T$$

where S and T are defined in FIG. 2*b* for continuous media. A close examination of Equation 4.29 reveals that the factor $(2\pi d/\lambda)$ is common to both numerator and denominator. Therefore, Equation 4.29 should reduce to Fresnel's formulae if terms of higher order than $(2\pi d/\lambda)$ are discarded. Accordingly,

$$\lim_{d\to 0}(S_1/T_1)=\begin{cases}(S/T)_n=\dfrac{iu_n}{iV(1-Q)^{1/2}+i(V-u_n)}=\dfrac{Q}{2(1-Q)^{1/2}+2-Q}=\dfrac{\sin\psi-v_{ij}\sin\psi_j}{\sin\psi+v_{ij}\sin\psi_j}\\ (S/T)_p=\dfrac{iu_n(\sin^2\psi-v_{ij}^{-2}\cos^2\psi)}{iV(1-Q)^{1/2}+i(V-u_p)}=\dfrac{Q_-}{2(1-Q)^{1/2}+2-Q_+}=\dfrac{\sin\psi_j-v_{ij}\sin\psi}{\sin\psi_j+v_{ij}\sin\psi}\end{cases}\quad(4.30)$$

where

$$\lim_{d\to 0} V=\lim_{d\to 0} u=0.$$

The last step of Equation 4.30*a* was obtained by multiplying the numerator and denominator by $1-[1-(1-v_{ij}^2)/\sin^2\psi]^{1/2}\}$ whereas for Equation 4.30*b* the numerator and denominator were divided by $[(1-v_{ij}^{-2}\cos^2\psi)^{1/2}+v_{ij}\sin\psi]$. It should be immediately recognized that Equation 4.30 is Fresnel's formulae for the reflection coefficient in continuous media[20] where $v_{ij}=(\epsilon_j/\epsilon_i)^{1/2}$ for $\mu_j/\mu_i=1$. To the best of the author's knowledge, Equation 4.30 marks the first time that Fresnel's formulae have been formally derived from the interference of waves scattered by each electron of the medium.

Substituting the conditions $V \ll 1$, $|\bar{u}| \ll 1$ into Equation 4.5, total reflection occurs for $[(V-u)/U]^2 \leq 1$. For these conditions,

$$[(V-u)/U]^2=\left\{\begin{matrix}[(V-u_n)/U_n]^2=[(1-Q/2)/Q/2)]^2[1-(V^2/3)(2R_{oT}/d)^2]^{-1}\\ [(V-u_p)/U_p]^2=[(1-Q_+/2)/(Q_-/2)]^2[1-(V^2/3)(2R_{oT}/d)^2]^{-1}\end{matrix}\right\}\leq 1\quad(4.31a)$$

The above inequality is satisfied by only the positive root for normal polarization and by only the negative root for parallel polarization. Accordingly, the effective critical angle ψ_c' for total internal reflection is given by

$$\sin\psi_o'=\begin{cases}\sin(\psi_o')_n=[\sin\psi_o][1-(V_o^2/24)(2R_{oT}/d)^2]\\ \sin(\psi_o')_p=[\sin\psi_o][1-(V_o^2/24)(\cos^4\psi_o)(2R_{oT}/d)^2]\end{cases}\quad V\ll 1,\ |\bar{u}|\ll 1\quad(4.31b)$$

where

$\sin\psi_c = (1-v_{ij}^2)^{1/2}$, $v_{ij} < 1$, = critical angle for continuous media

$V_c = (2\pi d/\lambda)\sin\psi_c$

$|S_1/T_1|^2 = 1$, $\sin\psi \leq \sin\psi_c'$ (For Conditions 4.13 only)

If $\sin\psi_c' = \sin(\psi_c + {}^{-}\psi_c)$ is expanded for $\Delta\psi_c << 1$, then

$$\psi_c' = \begin{cases} (\psi_c')_n = \psi_c - (V_c^2/24)(\tan\psi_c)(2R_{oT}/d)^2 \\ (\psi_c')_p = \psi_c - (V_c^2/24)(\tan\psi_c)(\cos^4\psi_c)(2R_{oT}/d)^2 \end{cases} \quad V<<1,\ |\overline{u}|<<1 \tag{4.32}$$

The effective critical angle ψ_c' has a weak dependency upon the atomic structure factor. The condition for total internal reflection, given by Equation 4.31, is a special case of a more general condition (Equation 4.5) for total reflection.

Substituting Conditions 4.13 into Equation 4.11, total transmission occurs for $U=0$ corresponding to $M = 0$. With reference to Equation 4.1, this condition reduces to

$$S_1/T_1 = 0, \begin{cases} v_{ij} = 1, \text{ normal and parallel polarization} \\ \psi = \cot^{-1} v_{ij} = \psi_{Brewster} \end{cases} \quad V<<1,\ |\overline{u}|<<1 \tag{4.33}$$

The condition $v_{ij} = 1$ is the case of an unbounded medium with no interfaces. The condition $\psi = \cot^{-1} v_{ij}$ is Brewster's angle for zero reflection of electromagnetic waves whose electric field is polarized parallel to the plane of incidence[5]. Brewster's condition is a special case of a more general condition, (Equation 4.11) for total transmission.

It is of interest to compare the reflection coefficient S_1/T_1, given by Equation 4.29 for Conditions 4.13, to the reflection coefficient S/T, given by Equation 4.30 for Conditions 4.14. Equation 4.29 can be expressed in a form analogous to that of Equation 4.30, namely

$$S_1/T_1 = \begin{cases} (S_1/T_1)_n = \dfrac{\sin\psi - (v_{ij}')_n \sin(\psi_j')_n}{\sin\psi + (v_{ij}')_n \sin(\psi_j')_n} \\ (S_1/T_1)_p = \dfrac{\sin(\psi_j')_p - (v_{ij}')_p \sin\psi}{\sin(\psi_j')_p + (v_{ij}')_p \sin\psi} \end{cases} \tag{4.34}$$

where $v_{ij}' = (\epsilon_j'/\epsilon_i)^{1/2}$ and ψ_j' are given by Equations 4.24 and **4.25** for Conditions 4.13. Equation 4.34 is expressed in terms of the effective refractive parameters corresponding to those defined in FIG. **10c**. In computing S_1/T_1 for Conditions 4.13, Equations 4.29 and 4.34 give identical results. For normal polarization, Equations 4.29 and 4.34 are given explicitly by

$$(S_1/T_1)_n = \frac{1 - i[(Q-1)(1+V^2Q^2/12) - (V^2Q^2/12)(2R_{oT}/d)^2]^{1/2}}{1 + i[(Q-1)(1+V^2Q^2/12) - (V^2Q^2/12)(2R_{oT}/d)^2]^{1/2}} \quad V<<1,\ |\overline{u}|<<1 \tag{4.35}$$

From Equation **4.35**,

$$|(S_1/T_1)_n|^2 - |(S/T)_n|^2 \begin{cases} <0, \sin\psi \leq \sin(\psi_c')_n,\ v_{ij} \text{ complex},\ v_{ij} \leq 1 \\ =0, \sin\psi \leq \sin(\psi_c')_n,\ v_{ij} \text{ real},\ v_{ij} \leq 1 \\ >0, \sin\psi > \sin(\psi_c')_n \text{ or } v_{ij}' > 1 \end{cases} \quad V<<1,\ |\overline{u}|<<1 \tag{4.36}$$

The percentage difference of the reflected intensities, for radiation of wavelength $\lambda = 5 \times 10^{-7}$m at normal incidence on fused quartz, is

$$[|S_1/T_1|^2 - |S/T|^2]/|S/T|^2 \approx 4(v_{ij}' - v_{ij})/(v_{ij}^2 - 1) = 1.4 \times 10^{-3}\%.$$

Consider now x-ray radiation incident, at grazing angles less than or equal to the effective critical angle $(\psi_c')_n$, on materials such as those in Table **1**. At x-ray wavelengths, the relative velocity $v_{ij} = v_i/v_j$ in continuous media is given by

$$v_{ij} = (1-\delta) - i\beta \tag{4.37}$$

where

δ = unit decrement of the index of refraction ($\delta << 1$)

β = absorption index

$\sin\psi_c = [1 - (\mathrm{Re}\, v_{ij})^2]^{1/2} = (2\delta)^{1/2}$

β/δ = absorption ratio

At the critical angle $(\psi_c')_n$ given by Equation 4.31*a*, Equation 4.35 reduces to

$$(S_1/T_1)_n = \frac{1 - i(\beta/\delta)[1 + (V_c^2/12) - (V_c^2/12)(2R_{oT}/d)^2]}{1 + i(\beta/\delta)[1 + (V_c^2/12) - (V_c^2/12)(2R_{oT}/d)^2]},\ \psi = (\psi_c')_n,\ \beta/\delta<<1,\ \delta<<1,\ V_c<<1 \tag{4.38}$$

It is noted from Equation 4.38 that the effective absorption ratio $(\beta/\delta)_n'$ has increased to an amount given by $(\beta/\delta)_n' = (\beta/\delta)\ \{1 + (V_c^2/12)[1 - (2R_{oT}/d)^2]\}$, $\psi = (\psi_c')_n$, $\beta/\delta << 1$, $\delta << 1$, $V_c << 1$, (4.39)

At lower grazing angles of incidence, the effective absorption ratio is larger and is given by

$$(\beta/\delta)_n' = (\beta/\delta)\left\{\frac{1 + (V_c^2/12)(3\sin^2\psi_c/\sin^2\psi) - 2[1 + (2R_{oT}/d)^2])}{1 + (V_c^2/12)((\sin^2\psi_c/\sin^2\psi) - [1 + (2R_{oT}/d)^2])}\right\} \beta/\delta<<1,\ \delta<<1,\ V<<1,\ |\overline{u}|<<1 \tag{4.40}$$

The effective absorption ratio in discrete media relative to that in continuous media is tabulated in Table **4** for some solid materials at x-ray wavelengths. Table **4** is consistent with the long-standing anomaly in x-ray physics[20, 22] that the measured absorption ratio is several per cent more than the calculated

TABLE 4.—RELATIVE EFFECTIVE ABSORPTION RATIO (NORMAL POLARIZATION) FOR SOME SOLID MATERIALS AT X-RAY WAVELENGTHS

$[(\beta/\delta)_n'/(\beta/\delta)]$

Material	$(\sin\psi)/(\sin\psi_o)$		
	1.	⅓	1/10
Beryllium, α	1.00009	1.0015	*1.018
Aluminum	1.00038	*1.0065	*1.074
Nickel	1.00093	*1.0158	*1.170
Gold	1.00230	*1.0390	*1.370
Quartz, α (SiO_2)	1.00057	*1.0097	*1.110

* Estimate only because the condition $(\bar{u}/\langle\langle 1$ is not satisfied.

NOTE.—Entries are calculated using Equation 4.40, the values of V_o given in Table 1, and the parameter $(2R_{oT}/d)=0$.

absorption ratio based on Fresnel's reflection coefficient for continuous media. A more complete discussion of this anomaly is given in the following subsection.

$V << 1,\ 0 \leq |\bar{u}| < \infty$

In the previous sub-section, the substitution of Conditions 4.13 for Conditions 4.14 led to relatively small deviations of the reflected and transmitted radiation from that predicted by existing theory for continuous media. Consider now the more general conditions

$$V << 1,\ 0 \leq |\bar{u}| < \infty \qquad (4.41)$$

For Conditions 4.41, the reflected and transmitted radiation can differ appreciably from that for continuous media.

From Equations 4.11 and 4.15, the angles $\psi_{b,M}$ for total transmission are determined exactly by the conditions

$$(U_{b,M})_n = (2\pi d/\lambda)(1/\sin\psi)(1/2)(1-v_{ij}^2)[1-(V^2/6)(2R_{oT}/d)^2] = M\pi$$

$$(U_{b,M})_p = (2\pi d/\lambda)(1/\sin\psi)(1/2)(1-v_{ij}^2)(\sin^2\psi - v_{ij}^{-2}\cos^2\psi)[1-(V^2/6)(2R_{oT}/d)^2] = M\pi \quad M = 0, \pm 1, \pm 2, \pm 3, \ldots \qquad (4.42)$$

Defining $V_c = (2\pi d/\lambda)(1-v_{ij}^2)^{1/2}$, Equations 4.42 reduce to

$$M=0, \begin{cases} (\psi_{b,0})_n \text{ is not defined} \\ (\psi_{b,0})_p = \psi_{Brewster} = \cot^{-1} v_{ij} \\ v_{ij}=1 \text{ (normal and parallel polarizations)} \end{cases} \qquad (4.42a)$$

$$M=\pm 1, \pm 2, \ldots, \begin{cases} (1-v_{ij}^2)/\sin^2(\psi_{b,M})_n \approx (2M\pi/V_o)^2 \text{ (normal polarization)} \\ (1-v_{ij}^2)/v_{ij}^2 \tan^2(\psi_{b,M})_p \approx (2M\pi v_{ij}/V_o)^2 \text{ (parallel polar).} \end{cases} \qquad (4.42b)$$

where $(V_c^2/12)(2R_{oT}/d)^2(V_c/M\pi)^2 << 1$ (normal polarization), and $(V_c^2/24)(2R_{oT}/d)^2(V_c/M\pi\ v_{ij}^2)^2 << 1$, $(1+v_{ij}^2)(V_c/M\pi v_{ij})^2 << 1$ (parallel polarization). Equation 4.42a was derived earlier (see Equation 4.33) and corresponds to total transmission for either Conditions 4.13 or the more restrictive Conditions 4.14 for continuous media. For normal polarization, Equation 4.42b is identical to the qualitative results derived earlier (see Equation 2.6). The propagation of electromagnetic waves at low grazing angles of incidence is a strong function of the polarization of the waves since $\sin(\psi_{b,M})_n/\sin(\psi_{b,M})_p \approx v_{ij}^2$, $M \mp 0$.

The general conditions for total reflection, subject to Conditions 4.41, are determined from Equations 4.5 and **4.15** which reduce, without approximation, to the conditions

$$(V/2)[1+(V^2/12)(2R_{oT}/d)^2] + M\pi \leq u_n \leq (V/2)[1+(V^2/12)(2R_{oT}/d)^2] + (2M+1)\pi/2,\ M=0, \pm 1, \pm 2, \ldots \text{ normal polarization}$$

$$(V/2)[1+(V^2 v_{ij}^4/12)(2R_{oT}/d)^2] + M\pi \leq u_n(\cos^2\psi)/v_{ij}^2 \leq (V/2)[1+(V^2 v_{ij}^4/12)(2R_{oT}/d)^2] + (2M+1)\pi/2,\ M=0, \pm 1, \pm 2 \ldots, \text{ parallel polarization} \qquad (4.43)$$

Equations 4.43 reduce to the following conditions for total reflection:

$$M=0, \begin{cases} 1+(V_o^2/12)(2R_{oT}/d)^2 \leq \sin^2\psi_o/\sin^2\psi \lesssim 1+(\pi/V_o)^2, \text{ normal polarization} \\ 1+(V_o^2/12)(\cos^4\psi_o)(2R_{oT}/d)^2 \leq \tan^2\psi_o/\tan^2\psi \lesssim 1+[(\pi\cos\psi_o)/V_o]^2, \text{ parallel polarization} \end{cases}$$

where $(2V_c/\pi)^2 << 1$ (normal polarization) and $(2V_c/\pi v_{ij})^2 << 1$ (parallel polarization) (4.43a)

$$M=\pm 1, \pm 2, \ldots, \begin{cases} 1+(2M\pi/V_o)^2 \lesssim (1-v_{ij}^2)/\sin^2\psi \lesssim 1+[(2M+1)\pi/V_o]^2, \text{ normal polarization} \\ 1+(2M\pi v_{ij}/V_o)^2 \lesssim (1-v_{ij}^2)/v_{ij}^2\tan^2\psi \lesssim 1+[(2M+1)\pi v_{ij}/V_o]^2, \text{ parallel polarization} \end{cases}$$

where $(V_c^2/12)(2R_{oT}/d)^2 << 1$, $[2V_c/(2M+1)\pi]^2 << 1$ (normal polarization) and $(V_c^2/12)(2R_{oT}/d)^2 << 1$, $[2V_c/(2M+1)\pi v_{ij}]^2 << 1$ (parallel polarization). (4.43b) The left-hand boundary of Equation 4.43a was derived earlier for Conditions 4.13 (see Equation 4.31). The right-hand boundary of Equation 4.43a indicates that total reflection does not occur at all grazing angles less than the critical angle ψ_c' for the case $v_{ij} < 1$. Equation 4.43b indicates that in phase-ordered media there are periodic regions of total reflection regardless of whether $v_{ij} < 1$ or $v_{ij} > 1$. Equations 4.43 differ in an important respect from the qualitative results derived earlier (see Equation 2.5). The qualitative theory predicted that total reflection should occur at discrete angles $\psi_{a,M}$ of zero bandwidth. The quantitative theory shows that total reflection occurs in discrete angular bands whose bandwidths include the discrete angles $\psi_{a,M}$.

If the angles $\psi_{\bar{a},M}$ and $\psi_{a,M}$ are defined as the angles corresponding to the left and right hand boundaries respectively of the total reflection regions of Equations 4.43a and 4.43b, then

$$\psi_{a,M} < \psi_{a,M} < \psi_{a,M}$$

where

$$\left.\begin{array}{l}(1-v_{ij}^2)/\sin^2 (\psi_{a,M})_n \approx 1+[(2M+1)\pi/V_c]^2 \\ (1-v_{ij}^2)/[v_{ij}^2 \tan^2 (\psi_{a,M})_p] \approx 1+[(2M+1)\pi v_{ij}/V_c]^2\end{array}\right\} \text{defines } \psi_{a,M}$$

$$\left.\begin{array}{l}(1-v_{ij}^2)/\sin^2 (\psi_{a,M})_n \approx [(M+1)\pi/V_c]^2 \\ (1-v_{ij}^2)/[v_{ij}^2 \tan^2 (\psi_{a,M})_p] \approx [(2M+1)\pi v_{ij}/V_c]^2\end{array}\right\} \text{defines } \psi_{a,M}$$

$$\left.\begin{array}{l}(1-v_{ij}^2)/\sin^2 (\psi_{a,M})_n \approx 1+(2M\pi/V_c \\ (1-v_{ij}^2)/[v_{ij}^2 \tan^2 (\psi_{a,M})_p] \approx 1+(2M\pi v_{ij}/V_c)^2\end{array}\right\} \text{defines } \psi_{a,M} \qquad (4.43c)$$

The percentage bandwidth $\Delta\psi_{\bar{a},M}/\psi_{\bar{a},M}$, of each total reflection region of Equations 4.43a and 4.43b, is given approximately by

$$\Delta\psi_{\bar{a},M}/\psi_{\bar{a},M} \approx \Delta\lambda_{\bar{a},M}/\lambda_{\bar{a},M} \approx 1/(2M+1)\ , \ M=0,\pm1,\pm2,\ldots \qquad (4.44)$$

It should be noted that $\psi_{\bar{a},0} = \psi_c'$ = effective critical angle of total reflection.

Unlike total reflection, total transmission in phase-ordered media does not occur in bands but at discrete angles $\psi_{b,M}$ given by Equation 4.42. The reflection coefficient S_1/T_1, for $\psi \approx \psi_{b,M}$ or $U \approx U_{b,M}$, is determined by expanding Equation 4.9 for the conditions $|U-U_{b,M}| << 1$, $V << 1$. Retaining only the first order terms of the Taylor series expansion of Equation 4.9,

$$S_1/T_1=\begin{cases}(S_1/T_1)_n \approx \dfrac{(Q_M)_n}{2[1-(Q_M)_n]^{1/2}+2-(Q_M)_n} \\ (S_1/T_1)_p \approx \dfrac{(Q_M)_p(\sin^2\psi - v_{ij}^{-2}\cos^2\psi)}{2[1-(Q_M)_p]^{1/2}+2-(Q_M)_p(\sin^2\psi+v_{ij}^{-2}\cos^2\psi)} \\ \qquad V \ll 1,\ |U-U_{b,M}| \ll 1 \qquad (4.45)\end{cases}$$

where

$$(Q_M)_n = (1-v_{ij}^2)[\csc^2\psi - \csc^2(\psi_{b,M})_n] = Q-(2M\pi/V_c)^2,\ M=0,\pm1,\pm2,\ldots.$$

$$(Q_M)_n = (1-v_{ij}^2)[\csc^2\psi - \csc^2(\psi_{b,M})_n] = Q - (2M\pi v_{ij}^2/V_c)^2,\ M=0,\pm1,\pm2,\ldots$$

Equations 4.45 are identical in form to Fresnel's equations for continuous media if the parameter Q_M is substituted for the parameter Q. For $M=0$, $Q_M=Q_0=Q$ and Equation 4.45 reduces to Fresnel's equations for continuous media given by Equation 4.30. Fresnel's equations are therefore a special case of Equation 4.45 which in turn is a special case of Equation 4.4. The intensity of the reflection coefficient,for angles of incidence approximately equal to $\psi_{b,M}$, is plotted in FIG. 9 for the case of normal polarization and Im $v_{ij}=0$. If the transmission bandwidth ΔQ_M is defined as the range of values of Q_M for which $|S_1/T_1|^2 \leq 0.5$, then it is seen in FIG. 4 that $(\Delta Q_M)_n = 33$. The angular and wavelength transmission bandwidths corresponding to $(\Delta Q_M)_n = 33$ are given by

$$(\Delta\psi_{b,M}/\psi_{b,M})_n = (\Delta\lambda_{b,M}/\lambda_{b,M})_n \approx (33/2)(V_c/2M\pi)^2,\ M=\pm1,\pm2,\ldots. \qquad (4.46)$$

Equation 4.46 was obtained by setting $\psi = \psi_{b,M} + \Delta\psi$ in Equation 4.45 for Q_M. For normal polarization, the angular transmission bandwidth is not defined for $M=0$ because $(\psi_{b,0})_n$ is not defined. For parallel polarization, the reflection coefficient and the transmission bandwidth are functions of the parameters v_{ij} and the angle ψ in addition to the parameter $(Q_M)_p$. Therefore it is easier to discuss the case of normal polarization. The transmission bandwidths for a free space-quartz interface is tabulated in Table 5 for the case of normal polarization and $M=1$. The transmission bandwidths of Table 5 are extremely narrow throughout the electromagnetic spectrum. The bandwidths are so narrow that they may be considered as line bandwidths. Whereas the theory for continuous media predicts that total reflection or near total reflection should occur at low grazing angles of incidence, the theory for discrete, phase-ordered media predicts that total transmission can occur at discrete wavelengths at certain discrete low grazing angles of incidence.

The extremely narrow transmission bands, defined by $|S_1/T_1|^2 \leq 0.5$, are part of much larger partial transmission regions defined by $|S_1/T_1|^2 < 1$. The regions of partial transmission are defined by

$$V_{\underline{a},M-1} + (2M-1)\pi \leq 2U \leq V_{a,M} + 2M\pi\ ,\ |S_1/T_1| < 1,\ V << 1 \quad (4.47)$$

where $V_{\underline{a},M-1}$ and $V_{a,M}$ correspond to the values of V for angles $\psi_{\underline{a},M-1}$ and $\psi_{\bar{a},M}$ respectively. In addition to the previously defined angles $\psi_{\bar{a},M}$, $\psi_{a,M}$, $\psi_{\underline{a},M}$, and $\psi_{b,M}$, it is convenient to define the angle $\psi_{d,M}$, near $\psi_{\underline{a},M-1}$ but in the transmission region, given by

$$(1-v_{ij}^2)/\sin^2(\psi_{d,M})_n = [2\ V_c/(2M+1)\pi]^2 \qquad (4.48)$$

$$(1-v_{ij}^2)/\tan^2(\psi_{d,M})_p = [2\ V_c/(2M+1)\pi v_{ij}]^2$$

TABLE 5.—TRANSMISSION BANDWIDTH FOR ELECTROMAGNETIC WAVES (NORMAL POLARIZATION) INCIDENT FROM FREE-SPACE ONTO QUARTZ AT THE ANGLE $\psi_{b,1}$ OF TOTAL TRANSMISSION

Region	Wavelength λ (m)	$(\psi_{b,1})_n$ (rad)	$(\Delta\psi_{b,1}/\psi_{b,1})_n$	$(\Delta\lambda_{b,1})_n$(m)	$(\Delta\psi_{b,1})_n$ (rad)
Microwave	1	1.37×10^{-9}	1.11×10^{-17}	1.11×10^{-17}	4.00×10^{-26}
	10^{-2}	1.37×10^{-7}	1.11×10^{-13}	1.11×10^{-15}	4.00×10^{-20}
	10^{-3}	1.37×10^{-6}	1.11×10^{-11}	1.11×10^{-14}	4.00×10^{-17}
Infrared	10^{-4}	6.12×10^{-6}	4.95×10^{-10}	4.95×10^{-14}	1.35×10^{-15}
	10^{-6}	6.12×10^{-4}	4.95×10^{-6}	4.95×10^{-12}	1.35×10^{-9}
Visible	5×10^{-7}	1.27×10^{-3}	1.99×10^{-5}	9.90×10^{-12}	1.13×10^{-8}
Ultraviolet	1.6×10^{-7}	9.20×10^{-3}	4.69×10^{-4}	7.20×10^{-11}	1.08×10^{-6}
X-ray	(1)	$1.31\times10^{-2}\times\sin\psi_c$	2.85×10^{-3}	$3.71\times10^{-8}\times\sin\psi_c$	$3.74\times10^{-5}\times\sin\psi_c$

NOTE.—$d=4.9\times10^{-10}$m, $v_{ij}=1.95$ (microwave), 1.5 (visible, infrared), 2.0 (ultraviolet).

From Equation 4.9, the intensity of the reflection coefficient for the angle $\psi_{d,M}$ is given by $|S_1/T_1|^2 \approx 1-2V_{d,M} = 1-4V_c^2/(2M+1)\pi$. If $v_{ij}^2 = 2.25$, $d = 4.9\times10^{-10}$m, and $\lambda = 5\times10^{-7}$m, then $\psi_{d,1} = 1.64\times10^{-3}$radians and $|S_1/T_1|^2 = 1-2.02\times10^{-5}$. For these same values of $\psi_{d,1}$ and $|S_1/T_1|^2$ in a clad optical fiber of length $L=10$ m and diameter $D=10^{-6}m$, the radiation would be reflected $(L/D)\psi_{d,1}$ times with a net transmission $|T/T_1|^2 = |S_1/T_1|^{(2L/D)\psi d,1} = 0.72$. The radiation loss resulting from the discrete nature of the cladding material has not been considered in previous studies of fiber optic transmission[23].

The intensity $|S_1/T_1|^2$ of the reflection coefficient for phase-ordered media subject to the conditions $V << 1$, $0/\leq|\bar{u}|<\infty$, $Im\ v_{ij}=0$, is sketched in FIG. **11** for grazing angles of incidence $0 < \psi \leq \pi/2$. As the grazing angle of incidence is decreased towards zero, the interference of the scattered waves changes periodically from constructive to destructive interference with a corresponding periodic change in the reflection coefficient from total transmission to total reflection. Total reflection or near total reflection occurs unless the phase difference **2**U of the forward scattered waves is almost exactly zero or a multiple of 2π radians. With reference to FIG. **9** and Equations 4.43 and 4.46, the percentage transmission bandwidth $2(\Delta U)_n/2\pi$, in each zone of phase difference $2U=2\pi$ radians, for normal polarization is given by

$$2(\Delta U)_n/2\pi \approx \begin{cases} [(32)^{1/2}+1](V_c/2\pi), & M=0 \\ (33/2)(V_c/2M\pi)^2, & M=\pm1, \pm2, \ldots \end{cases} \quad (4.49)$$

In FIG. **11**, the transmission bands are so narrow that they appear almost as discrete lines. The largest transmission bandwidth occurs for the ground zone $M=0$. The ground zone $M=0$ includes the condition $2|U|<< 1$ and therefore also includes the condition $|\bar{u}|<< 1$. Since the parameter $\bar{u}$ is a measure of the phase difference of the single-scattered forward waves between adjacent lattice planes, the zone number M specifies the multiple of 2π radians that is associated with the phase difference $2U$ between adjacent lattice planes of the forward-scattered waves. Present electromagnetic theory is included in the ground zone $M=0$. FIG. **11** clearly demonstrates that a general treatment of electromagnetic waves must also include the higher order zones of interference of the scattered forward waves.

The amplitude delay multiple-scattering coefficient X, in phase-ordered media subject to the conditions $V <<1, 0 \leqslant |\bar{u}| <\infty, Im\ v_{ij}=0$, is determined from Equation 4.3 by use of the approximations $u_n \approx U_n, u_p \approx -U_p$ for M 0. The amplitude delay X is tabulated in Table 6 as a function of $2U$ with the corresponding grazing angles ψ of incidence. The principal value of the argument of X is given in Table 6 rather than the principal value plus $2M\pi$ radians. For example, $X=-1$ is written as $\exp[(-1)^{M+1}i\pi]$ rather than $\exp[-i(2M-1)\pi]$. The reason for selecting the principal value is so that both $ln\ X$ and X are periodic functions of the angle of incidence ψ. The cyclical path traced by the vector X, as the angle of incidence is decreased towards zero, is shown in FIG. 8b with the corresponding values of the angle ψ given by Table 6. The direction of the path shown in FIG. **8**b corresponds to decreasing values of ψ if $v_{ij} < 1$ and to increasing values of ψ if $v_{ij} < 1$.

The effective relative velocity v_{ij}' and the effective direction of propagation ψ_j' are determined by direct substitution of $ln\ X$ into Equations 2.21 and 2.22. The refractive parameters v_{ij}' and ψ_j' are tabulated in Table 6 and plotted in FIGS. **13** and **14.** In phase-ordered media, the parameters v_{ij}' and ψ_j' are quasi-periodic functions of the angle of incidence ψ. For each value of U (and ψ)

TABLE 6.—ELECTROMAGNETIC PROPERTIES OF DISCRETE, PHASE-ORDERED MEDIA SUBJECT TO THE CONDITIONS $V<<1, 0\leq|\bar{u}|<\infty$, $Im\ v_{ij}=0$.

				$-\ln X=\gamma_{jz}'\cdot d$			
ψ	2U	$\left\lvert\frac{S_1}{T_1}\right\rvert^2$	X	$\alpha_{jz}'d$	$\beta_{jz}'d$	v_{ij}'	ψ_j'
$\psi_{a,M-1}$	$V_{a,M-1}+(2M-1)\pi$	1	$-1=\exp[(-1)^{M+1}i\pi]$	0	$(-1)^{M+1}\pi$	$[\cos^2(\psi_{a,M-1})+(\lambda\pi/2\pi d)^2]^{1/2}>>1$	$\approx\pi/2$.
$\psi_{d,M}$	$2V_{d,M}+(2M-1)\pi$	<1	$(-1)^{M+1}i=\exp[(-1)^{M+1}i\pi/2]$	0	$(-1)^{M+1}\pi/2$	$[\cos^2(\psi d,M)+(\lambda\pi/4\pi d)^2]^{1/2}>>1$	$\approx\pi/2$.
$\psi_{b,M}$	$2M\pi$	0	$\exp[(-1)^{M+1}iV_{b,M}]$	0	$(-1)^{M+1}V_{b,M}$	$\cot(\psi_{b,0})p^1$, M=0	$\psi_{b,M}$ M=0.
$\psi_{a,M}$	$V_{a,M}+2M\pi$	1	$1=\exp(0)$	0	0	$\cos\psi_{a,M}$	0.
$\psi_{a,M}$	$(2M+1)\pi$	1	$0=\exp(-\infty)$	∞	0	$i\infty$	Cut-off.
$\psi_{a,M}$	$V_{a,M}+(2M+1)\pi$	1	$-1=\exp[(-1)^{M+1}i\pi]$	0	$(-1)^{M+1}\pi$	$[\cos^2(\psi_{a,M})+(\lambda\pi/2\pi d)^2]^{1/2}>>1$	$\approx\pi/2$.

satisfying the condition $2U=2M\pi$, $v_{ij}'=1$ and $\psi_j' = \psi_{b,M}'M \approx 0$. For $M=0, |U|<<1, v_{ij}' \approx v_{ij}$ and $\psi_j' \approx \psi_j$. Phase-ordered media, at low grazing angles of incidence is highly birefringent since $(\psi_{b,M})_n/(\psi_{b,M})_p = v_{ij}^2$ for $M \neq 0$. The effective anisotropy of phase-ordered media at low grazing angles of incidence can be much larger than anisotropic crystals at angles perpendicular and parallel to the optic axis.

Thus far the explicit treatment of Conditions 4.41 has been concerned with the results for media with zero bulk absorption loss ($Im\ v_{ij}=0$). Two important examples, which are relevant to Conditions 4.41 and which involve media with absorption loss, are now considered.

As a first example, consider x-ray radiation incident from free space at grazing angles $\psi \leqslant \psi_c$. In Table 4 it is noted that some of the entries are only estimates of the effective absorption ratio $(\beta/\delta)_n'$ because the condition $|\bar{u}| << 1$ is not satisfied. For x-ray radiation incident on gold at an angle $\psi \approx (1/10)\psi_c$, Equation 4.40 estimates that the effective absorption ratio is 37 percent larger than the theoretical value for a continuous medium. For this case, $V_c = 1.66 \times 10^{-1}$ and $|\bar{u}| = (V_c/2)(\sin\psi_c/\sin\psi) = 0.83$. The condition $\bar{u}=0.83$ violates one of the Conditions 4.40. A more accurate value of the effective absorption ratio for normal polarization is determined by direct substitution of the parameters V and u_n into Equations 4.3 and 2.21. Let $\sin\psi_c = (2\delta)^{1/2} = 0.0412$ and $(\beta/\delta) = 0.241$ which are theoretical values based on the assumption of continuous media for this case [22]. Substituting these values into Equation 4.3, the amplitude delay $X_n = 0.821 \measuredangle -2.06°$. Substituting $ln\ X_n = |n|X_n + i\arctan(Im\ X_n/Re\ X_n) = 0.201 \measuredangle 190.4°$ into Equation 2.21, $(v_{ij}')_n = 1 - 1.31\ \delta - i\ 0.522\ \delta = 1 - \delta_n' - i\ \beta_n'$. The effective absorption ratio $(\beta/\delta)_n' = 0.522/1.31 = 0.398$. The effective absorption ratio relative to that for continuous media is given by

$$(\beta/\delta)_n' / (\beta/\delta) = 0.398/0.241 = 1.65, \text{ gold, x-ray radiation, } \psi=\psi_c/10 \quad (4.50)$$

The effective absorption ratio is 65 percent larger than the theoretical value for a continuous medium and 28 percent larger than the estimate of 37 percent given in Table 4 based on Equation 4.40. The experimental value of Hendrick for amorphous gold[22], $(\beta/\delta)_{exper.}$ / $(\beta/\delta) = 0.33/0.241 = 1.37$, was determined by the best fit of data to curves based on Fresnel's formula for continuous media. A true experimental value requires the best fit of data to curves based on Equations 4.3 and 2.21. The value given by Equation 4.50 is based on the structured lattice spacing a of Table 3 rather than the amorphous lattice spacing d_{av}. For amorphous gold, Conditions 2.7 may not be satisfied so that Equations 4.3 and 2.21 may not be applicable.

As a second example of media with bulk absorption loss and subject to Conditions 4.41, consider radiation incident from free space on non-magnetic metals at sufficiently low frequencies (DC — infrared) so that the complex relative permittivity $\epsilon_j/\epsilon_i = v_{ij}^2$ is given by [24]

$$v_{ij}^2 = (\mathrm{Re}\,\epsilon_j)/\epsilon_o - i\,\sigma/\omega\epsilon_o \approx 1 - i\,\sigma_{DC}/\epsilon_o\,\omega = 1 - i(\sigma_{DC}\lambda/2\pi)\,(\mu_o/\epsilon_o)^{1/2} \text{ non-magnetic metals, } \mathrm{Re}\,\epsilon_j \approx \epsilon_o,\ \sigma \approx \sigma_{DC} \quad (4.51)$$

where

σ_{DC} = DC conductivity of the metal (mhos/meter)

$(\mu_o/\epsilon_o)^{1/2}$ = characteristic impedance of free space = 377 ohms

$\omega = (2\pi/\lambda)(\epsilon_o\mu_o)^{-1/2}$ = radian frequency (rad/sec)

First consider the propagation in continuous metals (V=0, u=0) which is a special case of Conditions 4.41. From Equation 4.24, the propagation constant γ_j is given by

$$\gamma_j = i(2\pi/\lambda)[1 - i(\sigma_{DC}\lambda/2\pi)\sqrt{\mu_o/\epsilon_o}\,]^{1/2} \approx (1+i)[(\pi\sigma_{DC}/\lambda)\sqrt{\mu_o/\epsilon_o}\,]^{1/2} \text{ non-magnetic continuous metals, } (\sigma_{DC}\lambda/2\pi)(\mu_o/\epsilon_o)^{1/2} >> 1 \quad (4.52)$$

The wavelength λ_j of the radiation in the metal is given by

$$\lambda_j = 2\pi/\beta_j \approx [(4\pi\lambda/\sigma_{DC})\sqrt{\epsilon_o/\mu_o}\,]^{1/2}, \text{ Conditions 4.52} \quad (4.53)$$

The direction ψ_j of the transmitted radiation is given by

$$\psi_j = \cos^{-1}(v_{ij}^{-1}\cos\psi) \approx \cos^{-1} 0 = \pi/2 \text{ radians, Conditions 4.52} \quad (4.54)$$

The magnitude of the reflection coefficient is found from Equation 4.30 to be

$$|S/T|_n \approx \left|\frac{1-v_{ij}/\sin\psi}{1+v_{ij}/\sin\psi}\right| \approx 1-(2\sin\psi)/Re v_{ij} \approx 1-(4\sin\psi)[(\pi/\delta_{DC}\lambda)\sqrt{\epsilon_o/\mu_o}]^{1/2}$$

$$|S/T|_p \approx \left|\frac{1-v_{ij}\sin\psi}{1+v_{ij}\sin\psi}\right| \approx \left|\frac{1-(1-i)\sin\psi[(\delta_{DC}\lambda/4\pi)\sqrt{\mu_o/\epsilon_o}]^{1/2}}{1+(1-i)\sin\psi[(\delta_{DC}\lambda/4\pi)\sqrt{\mu_o/\epsilon_o}]^{1/2}}\right| \text{ Conditions 4.52} \quad (4.55)$$

The skin depth δ_{skin} of the current in a plane conductor is given by [25]

$$\delta_{skin} = (1/\alpha_j)_{\psi=\pi/2} \approx [(\lambda/\pi\sigma_{DC})\sqrt{\epsilon_o/\mu_o}\,]^{1/2}, \text{ Conditions 4.52} \quad (4.56)$$

The surface resistivity R_s of a plane conductor is given by [25]

$$R_s = (\sigma_{DC}\,\delta_{skin})^{-1} = [(\pi/\sigma_{DC}\lambda)\sqrt{\mu_o/\epsilon_o}\,]^{1/2}, \text{ Conditions 4.52} \quad (4.57)$$

Equations 4.51 – 4.57 are the formulae presently used to characterize the propagation of electromagnetic waves in non-magnetic metals. Although these formulae are strictly correct only for the conditions $V = \bar{u} = 0$, the results are almost identical, as shown by Equation 4.24, if the conditions $V << 1$, $|\bar{u}| << 1$ are satisfied. It has already been shown in Table 1 that the phase condition $V << 1$ is satisfied in metals for any angle of incidence and for wavelengths longer than ultraviolet wavelengths. The phase $\bar{u}$ is given by

$$u = (2\pi d/\lambda)(1/\sin\psi)(1/2)(1-v_{ij}^2) = i\,(\mu_o/\epsilon_o)^{1/2}\,(d\sigma_{DC}/2\sin\psi), \text{ Conditions 4.51} \quad (4.58)$$

The phase $\bar{u}$ is independent of wavelength assuming Conditions 4.51. The phase u is tabulated in Table 7 for several non-magnetic metals. From Table 7 it is evident that the phase u does not satisfy the phase condition $|\bar{u}|1/<<$ for the better conducting metals, even for radiation at normal incidence. Formulae 4.52 – 4.57 must therefore be in substantial error for discrete, phase-ordered metals. A correct treatment for phase-ordered metals now follows.

Consider radiation whose electric field intensity is polarized normal to the plane of incidence. Substituting Equation 4.58 into Equation 4.3a,

$$X_n = (\cos V) + i(\sin V)(\tan h|\bar{u}|) - \{[(\cos V) + i(\sin V)(\tan h|\bar{u}|)]^2 - 1\}^{1/2} \approx \exp(iV).\exp[-(1+i)(V\tan h|\bar{u}|)^{1/2}],\ V \ll 1, \text{ Conditions 4.51} \quad (4.59)$$

Substituting Equation 4.59 into Equation 2.21, the effective complex relative permittivity $(\epsilon_j'/\epsilon_i)_n = (v_{ij}')_n^2$ is given by

$$(v_{ij}')_n^2 = \cos^2\psi - (\lambda/2\pi d)^2[2iV(\tan h|\bar{u}|) - V^2 - 2iV(1+i)(V\tan h|\bar{u}|)^{1/2}] \approx 1 - i(\lambda/\pi d)(\sin\psi)(\tan h|\bar{u}|) = 1 - i(\lambda/\pi d)(\sin\psi)(\tan h)[(d\delta_{DC}/2\sin\psi)\sqrt{\mu_o/\epsilon_o}]V \ll 1, \text{ Conditions 4.51} \quad (4.60)$$

Equation 4.60 reduces to Equation 4.51 if $|\bar{u}| << 1$ since tanh $|\bar{u}| \approx |\bar{u}|$ for $|\bar{u}| << 1$. For the better conducting metals, tanh $|\bar{u}| \approx 1$ as shown in Table 7. Therefore the effective permittivity of discrete, phase-ordered metals is very weakly dependent upon the conductivity σ_{DC} and is strongly dependent upon the lattice spacing d. The effective relative phase velocity $(v_{ij}')_n$, given by Equation 4.60, has strong absorption loss for $(\pi d/\lambda) << \sin\psi \leq \pi/2$ and weak absorption loss for $0 \leq \sin\psi << (\pi d/\lambda)$. Specifically,

TABLE 7.—ELECTROMAGNETIC PROPERTIES OF PHASE-ORDERED, NON-MAGNETIC METALS AT LONG WAVELENGTHS

Metal	δ_{DC} (mhos/m.)	d(10^{-10} m.)	$\lvert\bar{u}\rvert \sin\psi$	Tanh $\lvert\bar{u}\sin\psi\rvert$	$\left[\frac{\tanh\lvert\bar{u}\sin\psi\rvert}{\lvert\bar{u}\sin\psi\rvert}\right]^{1/2}$	$R_s^1\cdot\lambda^{1/2}$ (ohms-m.$^{1/2}$)	$R_s\cdot\lambda^{1/2}$ (ohms-m.$^{1/2}$)
Silver	$614.\times10^7$	3.20	3.70	0.999	0.603	0.0265	0.0439
Copper (annealed)	5.80×10^7	2.84	3.10	0.996	0.566	0.0256	0.0452
Gold	4.10×10^7	3.20	2.47	0.986	0.636	0.0340	0.0534
Aluminum	3.54×10^7	3.16	2.11	0.971	0.681	0.0093	0.0577
Beryllium, α	2.25×10^7	2.48	1.05	0.782	0.683	0.0437	0.0641
Tungsten	1.81×10^7	3.12	1.06	0.786	0.859	0.0694	0.0807
Lead	0.48×10^7	3.86	0.35	0.336	0.993	0.154	0.155
Mercury	0.10×10^7	3.56	0.067	0.067	1.000	0.343	0.343

NOTE.—The above values of δ_{DC} and d are for amorphous, polycrystalline metals. A more correct tabulation should be based on the values for single-crystalline metals.

$$(v_{ij}')_n \approx \begin{cases} v_{ij}\ (\tan h|\bar{u}|/|\bar{u}|)^{1/2}, & (\pi d/\lambda) \ll \sin\psi \leq \pi/2 \\ 1 - i(\lambda/2\pi d)\sin\psi, & 0 \leq \sin\psi \ll (\pi d/\lambda) \end{cases}$$

$$\text{non-magnetic metal, } V \ll 1,\ (\mu_0/\epsilon_0)^{1/2}(\delta_{DC}\lambda/2\pi) \gg 1 \qquad (4.61)$$

Substituting $(v_{ij}')_n$ for v_{ij} in Equations 4.52 – 4.57,

$$(\gamma_j')_n \approx \begin{cases} \gamma_j(\tan h|\bar{u}|/|\bar{u}|)^{1/2}, & \pi d/\lambda) \ll \sin\psi \leq \pi/2 \\ i(2\pi/\lambda)[1 - i(\lambda/2\pi d)\sin\psi], & 0 \leq \sin\psi \ll (\pi d/\lambda) \end{cases}$$

$$\text{Conditions 4.61} \qquad (4.62)$$

$$(\lambda_j')_n \approx \begin{cases} \lambda_j(|\bar{u}|/\tan h|\bar{u}|^{1/2}, & (\pi d/\lambda) \ll \sin\psi \leq \pi/2 \\ \lambda, & 0 \leq \sin\psi \ll (\pi d/\lambda) \end{cases}$$

$$\text{Conditions 4.61} \qquad (4.63)$$

$$(\psi_j')_n \approx \begin{cases} \psi_j \approx \pi/2, & (\pi d/\lambda) \ll \sin\psi \leq \pi/2 \\ \psi, & 0 \leq \sin\psi \ll (\pi d/\lambda) \end{cases} \quad \text{Conditions 4.61} \qquad (4.64)$$

$$1 - /S_1/T_1/_n \approx \begin{cases} [1 - |S/T|_n](|\bar{u}|/\tan h\ |\bar{u}|)^{1/2}, & (\pi d/\lambda \ll \sin\psi \leq \pi/2 \\ 0, & 0 \leq \sin\psi \ll (\pi d/\lambda) \end{cases}$$

$$\text{Conditions 4.61} \qquad (4.65)$$

$$\delta_{skin}' = \delta_{skin}(|\bar{u}\sin\psi|\tan h|\bar{u}\sin\psi|)^{1/2}, \text{ Conditions 4.61} \qquad (4.66)$$

$$R_s' = R_s\ (\tan h|\bar{u}\sin\psi|/|\bar{u}\sin\psi|)^{1/2}, \text{ Conditions 4.61} \qquad (4.67)$$

For angles of incidence given by $(\pi d/\lambda) << \sin\psi \leq \pi/2$, the electromagnetic properties of discrete, phase-ordered metals are related to those of continuous metals by

$$(v_{ij}')_n/v_{ij} = (\gamma_j')_n/\gamma_j = \lambda_j/\lambda_j')_n = (1 - |S/T|_n)/(1 - |S_1/T_1|_n)$$
$$= (\tan h|\bar{u}|/|\bar{u}|)^{1/2} \qquad (4.68)$$

$$\delta_{skin}/\delta_{skin}' = R_s'/R_s = (\tan h|\bar{u}\sin\psi|/|\bar{u}\sin\psi|)^{1/2} \qquad (4.69)$$

The function $(\tanh|u|\ |u|)^{1/2}$ is plotted in FIG. 16. The function $(\tanh|\bar{u}\sin\psi|)/|u\sin\psi|$ is tabulated in Table 7 for several non-magnetic metals. The normalized surface resistivities $R_s'\ \lambda^{1/2}$ and $R_s\lambda^{1/2}$ are also tabulated in Table 7. The effective surface resistivity for discrete, phase-ordered metals is less than for continuous metals. The reflection coefficient is also less for discrete, phase-ordered metals than for continuous metals. Therefore, the power loss in discrete, phase-ordered conductors should be less for transmission lines (where the radiation is guided)and more for radiation pipes (where the radiation is relfected)than if the conductors were continuous metals. For radiation pipes, the discrepancy should become more apparent as the grazing angle of incidence is decreased. Most metals,of polycrystalline structure in which the crystals are randomly oriented, do not satisfy Conditions 2.7 and are therefore not phase-ordered. For such polycrystalline metals, neither the assumption of continuous media nor the assumption of discrete, phase-ordered media are appropriate. Experimetal results indicate that, for polycrystalline metals, the surface resistivity is larger [26] and the reflection coefficient is smaller [27, 28] than those calculated on the assumption of continuous media. The experimetal anomalies have previously been attributed to surface contaminations, surface roughness, and other experimetal errors. The surface resistivity of transmission lines could be substantially reduced, particularly if the conductors were coated with a metal whose phase order not only increased the skin depth but which also increased the conductivity.

For angles of incidence given by $0 \leq \sin\psi << (\pi d/\lambda)$, Equations 4.61 –4.65 indicate that electromagnetic radiation propagates with a loss tangent $\mathrm{Im}(v_{ij}')_n^2/Re(v_{ij}')_n^2 = (\lambda/\pi d)\sin\psi << 1$ and is not appreciably reflected at the air - metal interface. This result is in sharp contrast to the theory for continuous metals which predicts that electromagnetic radiation is strongly attenuated in metals and almost totally reflected at the air - metal interface.

Consider now the case where the electric field intensity is polarized parallel to the plane of incidence for radiation incident on a phase-ordered metal. Substituting Equation 4.58 into Equation 4.3*b*. it is recognized that simple expressions cannot be obtained for X_p at all angles of incidence. However, simple expressions have been obtained for angles of incidence given by

$$\psi:\begin{cases}\pi/2\\ \tan\ (\pi d/\lambda\sin\psi)\ll|\bar{u}|\sin^2\psi\ll 1\\ \cot\ (\pi d/\lambda\sin\psi)\ll|\bar{u}|\sin^2\psi\ll 1\end{cases}$$

$$\text{non-magnetic metal, } V\ll 1,\ (\mu_o/\epsilon_o)^{1/2}(\delta_{DC}\lambda/2\pi)\gg 1 \quad (4.70)$$

The condition $|\bar{u}|\sin^2\psi << 1$ corresponds to the condition $\psi << \pi/2$. For the conditions of Equation 4.70, Equation 4.3*b* reduces to

$$X_p=\begin{cases}X_n,\ \psi=\pi/2\\ \exp\ [-V\{[i\sqrt{\mu_o/\epsilon_o}(\delta_{DC}\lambda/2\pi\sin^2\psi)]-1\}^{1/2}]\ \tan\ (\pi d/\lambda\sin\psi)\ll|\bar{u}|\sin^2\psi\ll 1\\ -\exp\ \{-(1+i)[(|\bar{u}|/2)(\sin^2\psi)\tan\ (\pi d/\lambda\sin\psi)]^{-1/2}\}\ \cot\ (\pi d/\lambda\sin\psi)\ll|\bar{u}|\sin^2\ll 1\end{cases}$$

$$\text{conditions 4.70} \quad (4.71)$$

Substituting Equation 4.71 into Equation 2.21, the effective relative permittivity $(\epsilon_j'/\epsilon_i)_p=(v_{ij}')_p^2$ is given by

$$(v'_{ij})_p^2=\begin{cases}(v'_{ij})_n^2,\ \psi=\pi/2\\ v_{ij}^2,\ \tan\ (\pi d/\lambda\sin\psi)\ll|\bar{u}|\sin^2\psi\ll 1\\ (\lambda/2d)^2\{1-(i/\pi)[(2|\bar{u}|\sin^2\psi)\tan\ (\pi d/\lambda\sin\psi)]^{-1/2}\}\ \cot\ (\pi d/\lambda\sin\psi)\ll|\bar{u}|\sin^2\ll 1\end{cases}$$

$$\text{Conditions 4.70} \quad (4.72)$$

The propagation of electromagnetic waves in phase-ordered metals for parallel polarization may therefore be summarized as follows: for normal incidence, the properties are given by Equations 4.60–4.69; for a certain interval of grazing angles much smaller than $\pi/2$ radians, the properties are almost identical to those for continuous metals; for still smaller grazing angles of incidence, the radiation propagates with relatively little attenuation but with a phase velocity considerably less than for electromagnetic waves in free space.

$0\leq V<\infty,\ |\bar{u}|<<1$

The preceding sub-section was limited to electromagnetic wave-interactions which satisfy the condition $V << 1$. Some examples of electromagnetic wave-interactions which do not satisfy the condition $V << 1$ are: x-ray radiation incident at grazing angles larger than the critical angle of total reflection, optical radiation incident on fluids and organic matter of large lattice spacing *d*, and microwave radiation incident on low density gases and plasma. For the above example of x-ray radiation, the condition $|\bar{u}|<<1$ is also satisfied. Therefore, consider the phase conditions

$$0\leq V<\infty,\ |\bar{u}|<<1 \quad (4.73)$$

in media which are sufficiently phase-ordered so that Conditions 2.7 are also satisfied.

The general conditions for total reflection, subject to Conditions 4.73, is determined by substituting $V=N\pi+(V-N\pi)$, $N=0, 1, 2,\ldots,$ into Equation 4.5. Equation 4.5 reduces to

$$\cos^2[(V-N\pi)-u]/\cos^2U\ \overline{\geq}\ 1 \quad (4.74a)$$

If $|\bar{u}|<<1$, the above inequality can be satisfied only if $|V-N\pi|<<1$. Retaining the first two terms in the Taylor series expansion of Equation 4.74*a*,

$$\left[\frac{(V-N\pi)-u_n}{u_n(1/Z_A)F_T(\psi,h)}\right]^2\leq 1,\ \text{normal polarization,}$$

$$|\bar{u}|\ll 1, |V-N\pi|\ll 1 \quad (4.74b)$$

$$\left[\frac{(V-N\pi)-u_n(v_{ij}^{-2}\cos^2\psi+\sin^2\psi)}{u_n(\sin^2\psi-v_{ij}^{-2}\cos^2\psi)(1/Z_A)F_T(\psi,h)}\right]^2\leq 1,\ \text{parallel polarization}$$

For $N=0$, Equation 4.74*b* reduces to $\sin\psi\ \overline{\leq}\ \sin\psi_c'$ which is the case for total internal reflection given by Equation 4.31*a*. For $N=1, 2, \ldots,$ and $V=N\pi$, $V=V_{B,N}=(2\pi d/\lambda)\sin\psi_{B,N}$ = Bragg condition for total reflection. For $N \mp 0$, the conditions of Equation 4.74*b* reduce to $|1-v_{ij}^2|<<1$, $|V-V_{B,N}|<<1$. For $|V-V_{B,N}|<<1$, $\Delta\psi=\psi-\psi_{B,N}<<1$ and $\sin\psi\approx\sin\psi_{B,N}+\Delta\psi\cos\psi_{B,N}$. Equation 4.74*b* reduces to

$$\left[\frac{\Delta\psi_n(\sin 2\psi_{B,N})-(1-v_{ij}^2)}{(1-v_{ij}^2)(1/Z_A)F_T(\psi,h)}\right]^2\leq 1,$$

$$\left[\frac{\Delta\psi_p(\sin 2\psi_{B,N})-(1-v_{ij}^2)}{(\cos 2\psi_{B,N})(1-v_{ij}^2)(1/Z_A)F_T(\psi,h)}\right]^2\leq 1,\quad |\bar{u}|\ll 1,\ |V-V_{B,N}|\ll 1, 2,\ \ldots \quad (4.74c)$$

The range of angles $\Delta\psi_{B,N}$, for which total reflection occurs in the vicinity of the Bragg angles $\psi_{B,N}$, is found by setting the left side of Equation 4.74*c* equal to unity and solving the resulting quadratic equation for $\Delta\psi$. Accordingly,

$$\begin{aligned}(\Delta\psi_{B,N})_n&=(\psi_{D,N}-\psi_{B,N})[1\pm(\Delta\psi_{D,N})_n],\\ (\Delta\psi_{B,N})_p&=(\psi_{D,N}-\psi_{B,N})[1\pm(\Delta\psi_{D,N})_p],\end{aligned}\quad |\bar{u}|\ll 1,\ |V-V_{B,N}|\ll 1, N=1,2,\ \ldots \quad (4.75)$$

where

$$\psi_{D,N} = \psi_{B,N} + (1 - v_{ij}^2)/(\sin 2\,\psi_{B,N})$$

$$\Delta\psi_{D,N} = \begin{cases} (\Delta\psi_{D,N})_n = \psi_{D,N}(1/Z_A)F_T(\psi,h) \\ (\Delta\psi_{D,N})_p = \psi_{D,N}(\cos 2\psi_{B,N})(1/Z_A)F_T(\psi,h) \end{cases}$$

Equation 4.75 is identical to the results obtained by Darwin[1,12] for x-rays incident from free space onto the face of a single crystal. Total reflection occurs in bands centered about the angles $\psi_{D,N}$ with a total bandwidth $2\Delta\psi_{D,N}$. These results, which consider the multiple scattering by all the lattice planes of the medium, are in contrast to the results, $\psi = \psi_{B,N}$, $\Delta\psi = 0$, obtained by Bragg[7] who considered the single scattering by adjacent lattice planes. The intensity of the reflection coefficient, for any angle in the vicinity of the Bragg angles and for phase-ordered media with or without absorption loss, is found by expanding Equation 4.9 in a Taylor series subject to the conditions of Equation 4.75.

Accordingly,

$$|S_1/T_1|_n^2 = \left| \frac{(\psi - \psi_{D,N})}{(\Delta\psi_{D,N})_n} + \left\{ \left[\frac{(\psi - \psi_{D,N})}{(\Delta\psi_{D,N})_n} \right]^2 - 1 \right\}^{1/2} \right|^{-2}$$

$$|S_1/T_1|_p^2 = \left| \frac{(\psi - \psi_{D,N})}{(\Delta\psi_{D,N})_p} + \left\{ \left[\frac{(\psi - \psi_{D,N})}{(\Delta\psi_{D,N})_p} \right]^2 - 1 \right\}^{1/2} \right|^{-2}$$

$$|\bar{u}| \ll 1,\ |V - V_{B,N}| \ll 1,\ N-1, 2, \ldots \quad (4.76)$$

where $\psi_{D,N}$ and $\Delta\psi_{D,N}$ are given by Equation 4.75 and are complex numbers for media with absorption loss. The intensity of the reflection coefficient, in the vicinity of a Bragg angle, is sketched in FIG. **12** for the case of zero absorption loss. For x-rays of wavelength $\lambda = 0.708 \times 10^{-10} m$ which is incident on a perfect calcite crystal, $2(\Delta\psi_{D,1})_n = 1.47 \times 10^{-5}$ radians. For this example, the bandwidth for 50 percent reflection is 6 percent larger than the bandwidth for 100 percent reflection. This **3**db bandwidth is considerably larger than the 3 db transmission bandwidth at the angle $\psi_{b,1}$ of total transmission at a free space - quartz interface (see Table 5).

Although Equation 4.76 gives the reflection coefficient intensity for angles in the vicinity of the Bragg angles, one of the major anomalies of x-ray physics has been the inability to calculate the reflection coefficient at angles of incidence far removed from the Bragg angles. The explanation for this anomaly is that Equation 4.76 is derived from Equation 4.9 by imposing the restrictive conditions $|\bar{u}| << 1$, $|\overline{V - V_{B,N}}| << 1$. Furthermore, Equation 4.9 is based on the single-scattering coefficients a and t given by Equations 3.29 and 3.30 whereas Darwin derived Equation 4.76 by the use of his more restrictive single-scattering coefficients given by Equations 3.31. If only the restrictive condition $|\overline{V - V_{B,N}}| << 1$ is removed from Equation 4.76 then Equation 4.9 reduces to

$$|S_1/T_1|_n^2 = \left| \frac{U_n}{(1 - u_n^2/2)(\sin V) - u_n(\cos V) + \{1 - U_n^2 - [(1 - u_n^2/2)(\cos V) + u_n(\sin V)]^2\}^{1/2}} \right|^2$$

$$|S_1/T_1|_p^2 = \left| \frac{U_p}{(1 - u_p^2/2)(\sin V) - u_p(\cos V) + \{1 - U_p^2 - [(1 - u_p^2/2)(\cos V) + u_p(\sin V)]^2\}^{1/2}} \right|^2$$

$$|\bar{u}| \ll 1,\ 0 \le V \angle \infty \quad (4.77)$$

Equation 4.77 is applicable to any angle of incidence at x-ray wavelengths except for grazing angles much less than the critical angle ψ_c of total reflection. The x-ray anomaly associated with grazing angles less than the critical angle ψ_c has already been treated in the preceding sub-section. The intensity of the reflection coefficient is sketched in FIG. **16** for the conditions $|\bar{u}| << 1$, $0 \leq V < \infty$, $Im\ v_{ij} = 0$. In phase-ordered media which satisfy the condition $|\bar{\bar{u}}| << 1$, total transmission or almost total transmission occurs unless the phase difference $2V$ of the reflected scattered waves at adjacent lattice planes is almost zero or almost a multiple of 2π radians. In FIG. **16**, the reflection bands are so narrow that they appear almost as discrete lines. The largest reflection bandwidth occurs for the ground zone $N = 0$. In comparing FIG. **16** with FIG. **11**, it is noted that the zone number N is restricted to positive integers which satisfy the constraint $\sin \psi_{B,N} = (N\,\lambda/2d) \leq 1$ whereas the zone number M may be any integer between minus infinity and plus infinity. Present electromagnetic theory based on Maxwell's equations is included in the double ground zone (M=0, N=0).

For the condition $|\bar{u}| << 1$, the amplitude delay X, given by Equation 4.3, reduces to

$$X_n = \frac{(1 - u_n^2/2)(\cos V) + u_n(\sin V)}{1 - U_n^2/2} - \left\{ \left[\frac{(1 - u_n^2/2)(\cos V) + u_n(\sin V)}{1 - U_n^2/2} \right]^2 - 1 \right\}^{1/2}$$

$$X_p = \frac{(1 - u_p^2/2)(\cos V) + u_p(\sin V)}{1 - U_p^2/2} - \left\{ \left[\frac{(1 - u_p^2/2)(\cos V) + u_p(\sin V)}{1 - U_p^2/2} \right]^2 - 1 \right\}^{1/2}$$

$$|\bar{u}| \ll 1, 0 \le V \angle \infty, \quad (4.78)$$

For angles of incidence in the vicinity of the Bragg angles $\psi_{B,N}$, Equation 4.78 reduces to

$$X_n = (-1)^N \{[1 - (1/2)(V - N\pi - u_n)^2][1 + (1/2)U_n^2] - [U_n^2 - (V - N\pi - u_n)^2]^{1/2}\}$$

$$\approx (-1)^N \exp\{-[U_n^2 - (V - N\pi - u_n)^2]^{1/2}\}$$

$$\approx \exp(iN\pi) \exp\left\{(i2\pi d/\lambda)(1/2 \sin \psi_{B,N})(1 - v_{ij}^2)(1/Z_A)F_T(\psi,h)\left\{\left[\frac{(\psi - \psi_{D,N})}{(\Delta\psi_{D,N})_n}\right]^2 - 1\right\}^{1/2}\right\}$$

$$X_p = (-1)^N [1 - (1/2)(V - N\pi - u_p)^2][1 + (1/2)U_p^2] - [U_p^2 - (V - N\pi - u_p)^2]^{1/2}$$

$$\approx (-1)^N \exp\{-[U_p^2 - (V - N\pi - u_p)^2]^{1/2}\}$$

$$\approx \exp(iN\pi) \exp\left\{(i2\pi d/\lambda)(\cot \psi_{B,N})(1/2)(1 - v_{ij}^2)(1/Z_A)F_T(\psi,h)\left\{\left[\frac{(\psi - \psi_{D,N})}{(\Delta\psi_{D,N})_p}\right]^2 - 1\right\}^{1/2}\right\}$$

$$|\bar{u}| \ll 1,\ |V - V_{B,N}| \ll 1,\ N = 1, 2, \ldots \quad (4.79)$$

For the case where $Im\ v_{ij}=0$, the incident wave propagates with no attenuation outside the reflection bands and with attenuation inside the reflection bands. On the edges of the reflection bands $X = \exp(i\ N\pi)$. At the center of the reflection band, $X = \exp(i\ N\pi)\exp[-u_{B,N}(1/Z_A)F_T(\psi h)]$. Using these values of X, it can be shown that $S_1/T_1 = 1$ at the edge nearest the corresponding Bragg angle, $S_1/T_1 = i$ in the center of the reflection bands, and $S_1/T_1 = -1$ at the edge farthest from the corresponding Bragg angle. The effective relative velocity v_{ij}', for angles of incidence in the vicinity of the Bragg angles, is determined by sutstituting Equation 4.79 into Equation 2.21. Accordingly,

$$v_{ij}'=\begin{cases}(v_{ij}')_n\approx 1-(1/2)(1-v_{ij}^2)(1/Z_A)F_T(\psi,h)\left\{\left[\dfrac{(\psi-\psi_{D,N})}{(\Delta\psi_{D,N})_n}\right]^2-1\right\}^{1/2}\\[2ex] (v_{ij}')_p\approx 1-(1/2)(1-v_{ij}^2)(\cos\psi_{B,N})(1/Z_A)F_T(\psi,h)\left\{\left[\dfrac{(\psi-\psi_{D,N})}{(\Delta\psi_{D,N})_p}\right]^2-1\right\}^{1/2}\end{cases}$$

$$|\bar{u}|\ll 1, |V-V_{B,N}|\ll 1,\ N=1, 2, \ldots \quad (4.80)$$

In contrast to previous efforts [29,40], Equation 4.80 gives the correct effective relative phase velocity for angles of incidence in the vicinity of the Bragg angles.

$0 \leq V < \infty$, $0 \leq |u| < \infty$

The three separate sets of conditions, $V << 1$, $|\bar{u}| << 1$; $V << 1$, $0 \leq |\bar{u}| < \infty$; $0 \leq V < \infty$, $|\bar{u}| << 1$; include most electromagnetic wave-interactions occurring in practice. However, they involve only a small part of $\bar{u} - V$ phase space as shown in FIG. 17. Let an arbitrary coordinate in $\bar{u} - V$ phase space be denoted $(\bar{u}, V)$. The coordinate $(0, 0)$ corresponds to the case of continuous media which, prior to this paper, has been treated by Maxwell's field equations or equations reducible from Maxwell's field equations. The independent theory of this paper provides an exact treatment for any coordinate $(\bar{u}, V)$ including (o, o). In this sense, the theory of this paper provides a vastly more unified electromagnetic theory than had existed heretofore.

Although Fresnel's reflection and transmission coefficients are applicable only to the coordinate $(0, 0)$, the multiple-scattering reflection and transmission coefficients, given by Equations 2.14 and 2.25 respectively, are applicable to any coordinate $(\bar{u}, V)$. Equations 2.14 and 2.25 may also be expressed in a form analogous to Fresnel's equations if the effective refractive parameters (see Equations 2.15–2.23) are first determined as a functions of the single-scattering parameters Φ, a, and t. Accordingly,

$$S_1/T_1=\begin{cases}(S_1/T_1)_n=\dfrac{\sin\psi-(v'_{ij})_n\sin(\psi'_j)_n}{\sin\psi+(v'_{ij})_n\sin(\psi'_j)_n}\\[2ex] (S_1/T_1)_p=\dfrac{\sin(\psi'_j)_p-(v'_{ij})_p\sin\psi}{\sin(\psi'_j)_p+(v'_{ij})_p\sin\psi}\end{cases}\quad 0\leq\bar{u}<\infty,\ 0\leq V<\infty \quad (4.81)$$

$$\tau_1/T_1=\begin{cases}(\tau_1/T_1)_n=\dfrac{2\sin\psi}{\sin\psi+(v'_{ij})_n\sin(\psi'_j)_n}\\[2ex] (\tau_1/T_1)_p=\dfrac{2\sin\psi}{\sin(\psi'_j)_p+(v'_{ij})_p\sin\psi}\end{cases}\quad 0\leq\bar{u}<\infty,\ 0\leq V<\infty \quad (4.82)$$

where v_{ij}' and ψ_j' are given by Equations 2.21 and 2.22 respectively.

Explicit reductions of Equations 4.3 and 4.4 have been developed for the coordinates $(<< 1, << 1)$, $(<< 1, V)$, and $(\bar{u}, << 1)$. Examples of media for which the coordinates $\bar{u}$ and V can both be large are organic media, liquid crystals, ferroelectrics, ferromagnetics, and plasma. Equations 4.3 and 4.4 may also be reduced for such media provided that the media are sufficiently phase-ordered so that Conditions 2.7 are satisfied.

5.0 BOUNDED MEDIA

The present theory of electromagnetic waves in bounded media is based on the assumption of lumped, circuit elements and/or continuous media with distributed parameters. Ampere's circuital law[30] and Faraday's induction law[31] yield the telegrapher's transmission line equations for periodic, lumped, circuit elements. Maxwell's field equations[2] yield the wave equations for continuous media in bounded and unbounded space. This section develops the theory of wave propagation in bounded, phase-ordered media. Like the preceding section for semi-unbounded media, this section for bounded media is developed independently of Maxwell's equations or any circuit equations.

The concept of positive and negative travelling waves is useful in the treatment of bounded media. In Section 2.0 it was shown that the transmitted wave τ_r at the r th lattice plane is the resultant of the total forward wave T_r and the total reflected wave S_r at the r th lattice plane. The transmitted wave τ_r in a semi-unbounded medium is a travelling wave because T_r and S_r are travelling waves and their phases increase with increasing values of r at exactly the same rate and in the same direction. In a bounded medium, the total forward and reflected waves, which are generated in the positive direction at the first boundary, interfere with the total forward and reflected waves which are generated in the negative direction at the second boundary. The resultants of the positive and negative travelling waves in bounded media are standing waves; i.e. the total forward, total reflected, and transmitted waves are standing waves in bounded media. A positive wave with respect to a given interface ij is defined to be a travelling wave in medium j which is excited at the interface ij by a wave which is incident from medium i. A negative wave with respect to a given interface ij is defined to be a travelling wave in medium j which is excited at the interface ij by a wave which is incident from the same medium j. Examples of positive and negative waves in semi-unbounded media are shown in FIGS. 18a and 18b.

With reference to FIG. 18a the boundary conditions are identical to those that were treated in Section 2.0 (see FIG. 10a). The source of excitation is the wave of amplitude T_1 and wavelength λ_i which is incident from medium i at a grazing angle ψ_i at the lattice plane $r=1$

of medium j. The waves T_r and S_r at the r th lattice plane of medium j are positive waves with respect to the interface ij ar $r=1$. From Equations 2.11 – 2.14,

$$T_r = T_r^+ = T_1^+ (X_{ij})^{r-1}$$
$$S_r = S_r^+ = S_1^+ (X_{ij})^{r-1} \quad (5.1)$$

$$S_1^+/T_1^+ = S_r^+/T_r^+ = [a/(1-\Phi X_t)]_{ij}$$
$$= a_{ij}/(1-\Phi_{ij} X_{ij} t_{ij}) = \Gamma_{ij}' \quad (5.2)$$

where the subscripts ij indicates that the parameters are defined for a positive wave in a semi-unbounded medium j which is excited by a wave incident from medium i at the interface ij. The parameters Φ_{ij}, X_{ij}, a_{ij}, t_{ij} are given by Equations 2.2, 2.11, 3.29, and 3.30 respectively and are a function of the phases V_{ij} and $\bar{u}_{ij}$ given by

$$V = V_{ij} = (2\pi d_j/\lambda_i) \sin \psi_i$$
$$\bar{u} = \bar{u}_{ij} = (2\pi d_j/\lambda_i)(1/\sin \psi_i)(1/2)(1-v_{ij}^2) \quad (5.3)$$

where d_j is the lattice spacing of medium j and $v_{ij} = v_i/v_j = (\epsilon_j/\epsilon_i)^{1/2}$ = the wave phase velocity in medium i relative to that in medium j if media i and j were continuous. The reflection coefficient Γ_{ij}' denotes the multiple-scattering reflection coefficient in a semi-unbounded medium j which is excited by a wave incident from medium i at the interface ij. If medium j of FIG. 18a were continuous, then $\Gamma_{ij}' = \Gamma_{ij}$ where $\Gamma_{ij} = S/T$ is given by Equation 4.30. It is noted from Equation 4.30 that for normal incidence in continuous media, $(\Gamma_{ij})\ \psi_i = \pi/2 = -(\Gamma_{ji})\ \psi_j = \pi/2$. However, in discrete media, $(\Gamma_{ij}')\ \psi_i = \pi/2 \neq -(\Gamma_{ji}')\ \psi_j = \pi/2$. The reflection coefficient Γ_{ij}' is an explicit function of the lattice spacing in medium j whereas the reflection coefficient Γ_{ji}' is an explicit function of the lattice spacing in medium i.

In discrete, phase-ordered media the interface ij is defined by the spacing d_{ij} between the last lattice plane in medium i and the first lattice plane in medium j (see FIG. 18a). The spacing d_{ij} is generally not equal to either the lattice spacing d_i in medium i or the lattice spacing d_j in medium j. As the spacing between lattice planes in media i and j approach zero, the interface spacing d_{ij} approaches zero and the number of lattice planes in media i and j approach infinity. It is therefore proper to think of continuous media as consisting of an infinite number of lattice planes (or scattering centers) with zero spacing between lattice planes (or scattering centers). Even if medium j were free space, this concept is valid since the reflection coefficient Γ_{ij}', given by Equation 5.2, reduces to Fresnel's reflection coefficient in the limit of $d_j \rightarrow o$. The propagation of electromagnetic waves in a phase-ordered medium is therefore characterized by the interaction of a forward wave and a reflected wave at every lattice plane of the medium. With respect to medium j, the reflected wave is a positive travelling wave in the semi-unbounded medium j of FIG. 18a. Its phase increases in the same direction and at the same rate as the forward wave. Its amplitude is of the same physical dimension, namely, electric field intensity, as the forward wave. These self-consistent concepts of electromagnetic wave propagation in media are independent (but not necessarily inconsistent with) Maxwell's field equations which assume orthogonal electric and magnetic fields that propagate by mutual interaction as a forward wave in the positive direction or as a reflected wave in the negative direction.

With reference to FIG. 18b, the boundary conditions are similar to those of FIG. 18a except that the propagation in medium i is of interest rather than in medium j. The given excitation in medium i is a resultant transmitted wave of amplitude τ_1^+ which is incident with an effective wavelength λ_i' (and an effective phase velocity v_i') from medium i at an effective grazing angle ψ_i' at the first lattice plane of medium j. The lattice planes in the semi-unbounded medium i are numbered from the interface ij with $r=1$ corresponding to the first lattice plane of medium j and with $r=2$ corresponding to the first lattice plane of medium i. How the incident wave was generated at the lattice plane $r=1$ is not specified. Therefore, it is not possible to explicitly express λ_i', v_i', and ψ_i' as a function of the lattice spacing d_i in a semi-unbounded medium i. The parameters λ_i' and v_i' are related by $\lambda_i' = 2\pi v_i'/\omega$ where ω is the radian frequency which is independent of the lattice spacing d_i. It is also not possible to explicitly express τ_1^+ as the resultant of a forward and reflected wave in the semi-unbounded medium i. The incident wave τ_1^+ may be treated as a positive wave in medium i since its source of excitation is implied to be external to medium i at $r=\infty$. The incident wave τ_1^+ excites a wave of amplitude τ_1^- at the interface at $r=1$. The wave τ_1^- is a negative travelling wave in medium i since by definition a negative travelling wave in a given medium is one which is excited by a wave incident from the same medium. If it is assumed that λ_i' and ψ_i' are equal to the respective parameters $\lambda_i(=2\pi v_i/\omega)$ and ψ_i defined in FIG. 18a, then the boundary conditions of FIG. 18b are identical to those of FIG. 18a. For these assumptions, the amplitude τ_1^- is given by

$$\tau_1^-/\tau_1^+ = \Gamma_{ij}' \ ; \ \lambda_i' = \lambda_i,\ \psi_i' = \psi_i,\ v_i' = v_i \quad (5.4)$$

where Γ_{ij}' is defined by Equation 5.2. The reflection coefficient Γ_{ij}' and the phase parameters V_{ij} and u_{ij} of Equation 5.3 are therefore explicit functions of the lattice spacing d_j and implicit functions of the lattice spacing d_i and the phase velocity v_i'. Since the phase velocity v_i' is indeterminate, it is expedient to approximate v_i' by the known phase velocity v_i for a continuous medium i. This is the approximation that has been made in Sections 3.0 – 5.0 in characterizing the propagation in medium i. If it is assumed that the interfacing spacing d_{ij} is equal to the lattice spacing d_i and that the incident wavelength α_i' at $r=1$ is equal to the effective wavelength in medium i at $r=2$, then the positive and negative travelling waves in medium i at the lattice plane r are given by

$$\tau_r^+ = \tau_1^+ (X\infty_i)^{-(r-1)}$$
$$\tau_r^- = \tau_1^- (X\infty_i)^{(r-1)} \quad (5.5)$$

where $\overline{X_{\infty i}} = \exp(-i\ 2\ \pi d_i/\lambda_{iz}') = \exp[-i2\pi d_i(\sin\psi_i')/\lambda_i'] = \exp(-\gamma_i' d_i \sin\psi_i') \approx \exp(-\gamma_i d_i \sin\psi_i)$.

The treatment of positive and negative waves in semi-unbounded media can be applied to the treatment of bounded media. With reference to FIG. 18c, medium j is of finite extent with m lattice planes and bounded by medium i at $r=1$ and by medium k at $r=m+1$. The source of excitation is the wave of amplitude T_1 and wavelength λ_i which is incident from medium i at a grazing angle ψ_i at the lattice plane $r=1$ of medium j. The total forward wave T_r and the total reflected wave S_r at the r th lattice plane of medium j are standing waves and are the resultant of positive and negative travelling waves with respect to the interface ij. The total forward wave T_r and the total reflected wave S_r are therefore given by (compare with Equation 2.11)

$$T_r = T_1^+(X_{ij})^{r-1} + T_1^-(X_{ij})^{-(r-1)} = T_r^+ + T_r^-,$$
$$S_r = S_1^+(X_{ij})^{r-1} + S_1^-(X_{ij})^{-(r-1)} = S_r^+ + S_r^-,$$
$$1 \le r \le m+1 \quad (5.6)$$

where T_1^+, T_1^-, S_1^+, S_1^- are wave amplitudes which are to be determined by satisfying boundary conditions at $r=1$ and $r=m+1$. The incident wave T_1 is the source of excitation of the positive waves T_1^+ and S_1^+ at the interface ij. In turn, the positive waves T_{m+1}^+ and S_{m+1}^+ are the sources of excitation of the negative waves T_{m+1}^- and S_{m+1}^- at the interface jk. For $r=1$ in Equation 5.6, the wave amplitude T_1^+ is related to the incident wave amplitude T_1 by $T_1 = T_1^+ + T_1^-$. From the Equation 5.2, $S_1^+ = \Gamma_{ij}'T_1^+$. The wave amplitude S_1^- results from the positive wave T_1^+ being transmitted from $r=1$ to $r=m+1$, being reflected as a negative wave with respect to the boundary at $r=m+1$, and then being transmitted back to $r=1$. Accordingly, $S_1^- = T_1^+(X_{ij})^m\Gamma_{jk}'(X_{ij})^m$. The wave amplitude T_1^- results from the positive wave S_1^+ being transmitted from $r=1$ to $r=m+1$, being reflected as a negative wave with respect to the boundary at $r=m+1$, and then being transmitted back to $r=1$. Accordingly, $T_1^- = S_1^+(X_{ij})^m\,\Gamma_{jk}'(X_{ij})^m$. The boundary conditions at $r=1$ and $r=m+1$ are therefore characterized by four simultaneous equations with four unknowns and are given by

$$\begin{aligned} &T_1^+ + T_1^- = T_1 \\ &S_1^+ = \Gamma_{ij}'T_1^+ \\ &S_1^- = \Gamma_{jk}'(X_{ij})^{2m}T_1^+ \\ &T_1^- = \Gamma_{jk}'(X_{ij})^{2m}S_1^+ \end{aligned} \quad (5.7)$$

Solving Equations 5.7 for the unknown wave amplitudes,

$$\begin{aligned} &T_1^+/T_1 = 1/[1+\Gamma_{ij}'\Gamma_{jk}'(X_{ij})^{2m}] \\ &S_1^+/T_1 = \Gamma_{ij}'/[1+\Gamma_{ij}'\Gamma_{jk}'(X_{ij})^{2m}] \\ &T_1^-/T_1 = \Gamma_{ij}'\Gamma_{jk}'(X_{ij})^{2m}/[1+\Gamma_{ij}'\Gamma_{jk}'(X_{ij})^{2m}] \\ &S_1^-/T_1 = \Gamma_{jk}'(X_{ij})^{2m}/[1+\Gamma_{ij}'\Gamma_{jk}'(X_{ij}')^{2m}] \end{aligned} \quad (5.8)$$

Substituting Equations 5.8 into Equations 5.5,

$$T_r = \frac{T_1(X_{ij})^{r-1}}{1+\Gamma_{ij}'\Gamma_{jk}'(X_{ij})^{2m}}\,[1+\Gamma_{ij}'\Gamma_{jk}'(X_{ij})^{2(m-r+1)}]$$

$$S_r = \frac{T_1\Gamma_{ij}'(X_{ij})^{r-1}}{1+\Gamma_{ij}'\Gamma_{jk}'(X_{ij})^{2m}}\,[1+(\Gamma_{jk}'/\Gamma_{ij}')(X_{ij})^{2(m-r+1)}]$$

$$1 \le r \le m+1 \quad (5.9)$$

From Equation 2.24, the resultant transmitted positive wave τ_r^+ and negative wave τ_r^- are given by

$$\tau_r^+ = \begin{cases} (\tau_r^+)_n = [(T_1^+)_n + (S_1^+)_n](X_{ij})^{-(r-1)} \\ (\tau_r^+)_p = [(\sin\psi_i)/\sin(\psi_j')_p][(T_1^+)_p + (S_1^+)_p](X_{ij})^{r-1} \end{cases}$$
$$\tau_r^- = \begin{cases} (\tau_r^-)_n = [(T_1^-)_n + (S_1^-)_n](X_{ij})^{-r-1} \\ (\tau_r^-)_p = [(\sin\psi_i)/\sin(\psi_j')_p][(T_1^-)_p + (S_1)^-_p](X_{ij})^{-(r-1)} \end{cases}, \quad 1 \le r \le m+1 \quad (5.10)$$

The positive wave τ_r^+ propagates in medium j with an effective wavelength λ_j' and an effective angle of refraction ψ_j'. The wave τ_r^+ is incident at an angle of incidence ψ_j' at the interface jk where it excites the negative wave τ_r^-. The negative wave τ_r^- propagates in medium j with a wavelength λ_j' at an angle of reflection ψ_j'. The resultant amplitude τ_r of the standing wave at the r th lattice plane of medium j is given by

$$\tau_r = \tau_r^+ + \tau_r^- = \begin{cases} (\tau_r)_n = (S_r)_n + (T_r)_n \\ (\tau_r)_p = [(\sin\psi_i/\sin(\psi_j')_p][(S_r)_p + (T_r)_p \end{cases} \quad (5.11)$$

In analogy to the reflection coefficient defined by Equation 5.4 as the ratio of the resultant negative and positive waves at the interface of two semi-unbounded media, the reflection coefficient $\Gamma'(r)$ at the r th lattice plane of a bounded or unbounded medium is defined as

$$\Gamma'(r) = \tau_r^-/\tau_r^+ \quad (5.12)$$

It should be noted that in a bounded medium the reflection coefficient $\Gamma'(r)$ is not equal to S_r/T_r, S_r^+/T_r^+, or S_r^-/t_r^-. The reflection coefficient at the 1 st lattice plane of the bounded medium j of FIG. **18***c* found from Equations 5.9, 5.10, and 5.12 to be

$$\Gamma'(1) = [\Gamma_{ij}' + \Gamma_{jk}'(X_{ij})^{2m}]/[1 + \Gamma_{ij}'\,\Gamma_{jk}'\,(X_{ij})^{2m}] \quad (5.13)$$

If media i, j, k are continuous, then Equation 5.13 reduces to the well-known result of interference optics[32] given by

$$\Gamma'(r) = \Gamma'(z+1) = \Gamma'(1) = \Gamma(1) = \frac{[\Gamma_{ij} + \Gamma_{jk}\exp(-2\gamma_j\sin\psi \ldots w)]_j}{[1+\Gamma_{ij}\Gamma_{jk}\exp(-2\gamma_j\sin\psi_j w)]}\,d_i = d_j = d_k = 0 \quad (5.14)$$

where

$w = m\,d_j$ = thickness of medium j assuming that the interface spacing d_{jk} is included as part of medium j. The reflection coefficient at the $(m+1)$ th lattice plane is also found from Equations 5.9, 5.10, and 5.12 and reduces to

$$\Gamma'(m+1) = \tau_{m+1}^-/\tau_{m+1}^+ = \Gamma_{jk}' \quad (5.15)$$

The positive and negative waves incident and reflected respectively at an angle of incidence ψ_j' at the 1 st lattice plane of medium k are given by Equation 5.10 for $r=m+1$. In analogy to Equation 5.11 for the resultant transmitted wave in medium j, the resultant transmitted wave τ_k in medium k at its first lattice plane is given by

$$\tau_k = \begin{cases} (\tau_k)_n = (\tau_{m+1}^+)_n[1+(\Gamma_{jk}')_n] \\ (\tau_k)_p = [1/\sin(\psi_k')_p]\sin(\psi_j')_p(\tau_{m+1}^+)_p[1+(\Gamma_{jk}')_p] \end{cases} \quad (5.16)$$

where $\psi_k' = \cos^{-1}(v_{jk}^{-1}\cos\psi_j')$ = effective angle of refraction in medium k. Equations 5.9 – 5.16 are functions of Γ_{ij}' and Γ_{jk}' which are given by

$$\begin{aligned} &\Gamma_{ij}' = a_{ij}/(1-\Phi_{ij}X_{ij}t_{ij}) \\ &\Gamma_{jk}' = a_{jk}/(1-\Phi_{jk}X_{jk}t_{jk}) \end{aligned} \quad (5.17)$$

$$t_{ij}=\begin{cases}(t_{ij})_n=\cos\,[(u_n)_{ij}(1/Z_A)_jF_T(\psi_i,\,h)_j\ (\text{sign}\cos 2\psi_i)]\cdot\exp\,[i(u_n)_{ij}]\\(t_{ij})_p=\cos\,[(u_n)_{ij}(1/Z_A)_jF_T(\psi_i,\,h)_j\ (\cos 2\psi_i)]\cdot\exp\,[i(u_n)_{ij}]\end{cases}$$

$$t_{jk}=\begin{cases}(t_{jk})_n=\cos\,[(u_n)_{jk}(1/Z_A)_kF_T(\psi_j',\,h)_k\ (\text{sign}\cos 2\psi_j']\cdot\exp\,[i(u_n)_{jk}]\\(t_{jk})_p=\cos\,[(u_n)_{jk}(1/Z_A)_kF_T(\psi_j',\,h)_k\ (\cos 2\psi_j')]\cdot\exp\,[i(u_n)_{jk}]\end{cases}$$

$$1\le m\ll 10 \quad (5.23)$$

The parameters Φ_{ij} and Φ_{jk} are defined by Equation 2.2 as

$$\Phi_{ij}=\exp\,(-iV_{ij})=\exp\,[-i(2\pi d_j/\lambda_i)\sin\psi_i]$$
$$\Phi_{jk}=\exp\,(-iV_{jk})=\exp\,[-i(2\pi d_k/\lambda_j')\sin\psi_j'] \quad (5.18)$$

where

$\sin\psi_j' = [1-(v_{ij}')^{-2}\cos^2\psi_i]^{1/2}$

$\lambda_j' = \lambda_i/v_{ij}'$

$v_{ij}' = [\cos^2\psi_i - (\lambda_i/2\pi d_j)^2(ln\ X_{ij})^2]^{1/2}$, $v_{jk}' = [\cos^2\psi_j' - (\lambda_j'/2\pi d_k)^2(ln\ X_{jk})^2]^{1/2}$

The parameters X_{ij} and X_{jk} are defined by Equations 2.10 and 2.11 as

$$X_{ij}=A_{ij}\pm(A_{ij}^2-1)^{1/2},\ A_{ij}=(1/2\Phi_{ij}t_{ij})[1+(\Phi_{ij})^2(t_{ij}^2-a_{ij}^2)]$$
$$X_{jk}=A_{jk}\pm(A_{jk}^2-1)^{1/2},\ A_{jk}=(1/2\Phi_{jk}t_{jk})[1+(\Phi_{jk})^2(t_{jk}^2-a_{jk}^2)]$$
$$(5.19)$$

If the number m of lattice planes in medium j is sufficiently large ($m >> 1$) so that the parameter $v_{ij} = (\epsilon_j/\epsilon_i)^{1/2}$ is well defined, then the single-scattering coefficients a_{ij} and a_{jk} are defined by Equation 3.30 as

$$a_{ij}=\begin{cases}(a_{ij})_n=i\sin\,[\overline{u}_{ij}(1/Z_A)_jF_T(\psi_i,\,h)_j]\cdot\exp\,(i\overline{u}_{ij})\\(a_{ij})_p=i\sin\,[\overline{u}_{ij}(1/Z_A)_jF_T(\psi_i,\,h)_j(\sin^2\psi_i-v_{ij}^{-2}\cos^2\psi_i)]\cdot\exp\,[i\overline{u}_{ij}(v_{ij}^{-2}\cos^2\psi_i+\sin^2\psi'_j)]\end{cases}$$
$$a_{jk}=\begin{cases}(a_{jk})_n=i\sin\,[\overline{u}_{jk}(1/Z_A)_kF_T(\psi'_j,\,h)_k]\cdot\exp\,(i\overline{u}_{jk})\\(a_{jk})_p=i\sin\,[\overline{u}_{jk}(1/Z_A)_kF_T(\psi'_j,\,h)_k(\sin^2\psi'_j-v_k^2v'^{-2}_j\cos^2\psi'_j)]\cdot\exp\,[i\overline{u}_{jk}(v_{k2}v'^{-2}_j\cos\psi'_j+\sin^2\psi'_j]m\gg 1\end{cases}$$
$$(5.20)$$

The single-scattering coefficients t_{ij} and t_{jk} are defined by Equation 3.30 as

$$t_{ij}=\begin{cases}(t_{ij})_n=\cos\,[\overline{u}_{ij}(1/Z_A)_jF_T(\psi_i,\,h)_j]\cdot\exp\,(i\overline{u}_{ij})\\(t_{ij})_p=\cos\,[\overline{u}_{ij}(1/Z_A)_jF_T(\psi_i,\,h)_j(\sin^2\psi_i-v_{ij}^{-2}\cos^2\psi_i)]\cdot\exp\,[i\overline{u}_{ij}(v_{ij}^{-2}\cos^2\psi_i+\sin^2\psi_i)]\end{cases}$$
$$t_{jk}=\begin{cases}(t_{jk})_n=\cos\,[\overline{u}_{jk}(1/Z_A)_kF_T(\psi'_j,\,h)_k]\cdot\exp\,(i\overline{u}_{jk})\\(t_{jk})_p=\cos\,[\overline{u}_{jk}(1/Z_A)_kF_T(\psi'_j,\,h)_k(\sin^2\psi'_j-v_k^2v'^{-2}_j\cos^2\psi^{2\prime})]\cdot\exp\,[i\overline{u}_{jk}(v_k^2v'^{-2}_j\cos^2\psi'_j+\sin^2\psi'_j)]m\gg 1\end{cases}$$
$$(5.21)$$

where $\overline{u}_{ij} = (2\pi d_j/\lambda_i)(1/2\ \sin\psi_i)(1-v_{ij}^2)$, $\overline{u}_{jk} = (2\pi d_k/\lambda_j')(1/2\ \sin\psi_j')(1-v_j'^2v_k^{-2})$.

If the number m of lattice planes in medium j is small ($1 \le m << 10$) so that it is not meaningful to define a permittivity ϵ_j, then the single scattering coefficients a_{ij} and a_{jk} are defined by Equation 3.18 as

$$a_{ij}=\begin{cases}(a_{ij})_n=i\sin\,[(u_n)_{ij}(1/Z_A)_jF_T(\psi_i,\,h)_j\ (\text{sign}\cos 2\psi_i]\cdot\exp\,[i(u_n)_{ij}]\\(a_{ij})_p=i\sin\,[(u_n)_{ij}(1/Z_A)_jF_T(\psi_i,\,h)_j(\cos 2\psi_i)]\cdot\exp\,[i(u_n)_{ij}]\end{cases}$$
$$a_{jk}=\begin{cases}(a_{jk})_n=i\sin\,[(u_n)_{jk}(1/Z_A)_kF_T(\psi_j',\,h)_k\ (\text{sign}\cos 2\psi_j')]\cdot\exp\,[i(u_n)_{jk}]\\(a_{jk})_p=i\sin\,[(u_n)_{jk}(1/Z_A)_kF_T(\psi_j',\,h)_k\ (\cos 2\psi_j')]\cdot\exp\,[i(u_n)_{jk}]\end{cases}$$
$$1\le m\ll 10 \quad (5.22)$$

where

$(u_n)_{ij} = -(G_n\ Z_A\ M_o)_j\lambda_i/\sin\psi_i$

$(u_n)_{jk} = -(G_n\ Z_A\ M_o)_k\lambda_j'/\sin\psi_j'$

G_n = function related to the electronic or atomic polarization of the medium (see Equation 3.6)

M_o = number of atoms (or molecules) per unit area in a single lattice plane of the medium.

The single-scattering coefficients t_{ij} and t_{jk} are defined by Equation 3.19 as

Equations 5.9 – 5.23 give the general properties of electromagnetic waves in bounded, phase-ordered media.

If medium j and medium k are discrete media, then the reflection coefficient Γ_{jk}' is an explicit function of the atomic structure factor and lattice spacing in both media j and k. The reflection coefficient Γ_{ij}' is an explicit function of the atomic structure factor and lattice spacing of medium j only (because λ_i and ψ_i are unspecified functions of the lattice spacing and atomic structure factor of medium i). If media i and k are identical and continuous, then

$$v_{jk}'=v_j'/v_k'=v_j'/v_i=1/v_{ij}'$$
$$\cos\psi_k'=(v_k'/v_j')\cos\psi_j'=[(v_j'/v_i)\cos\psi_i](v_i/v_j')=\cos\psi_i$$
$$\text{or } \psi_k'=\psi_i v_k=v_i, d_i=d_k=0 \quad (5.24)$$

The reflection coefficient Γ_{ij}' may be written in the form given by Equation 4.34 where $\sin\psi = \sin\psi_i$. Similarly, the reflection coefficient Γ_{jk}' may be written as

$$\Gamma_{jk}'=\begin{cases}(\Gamma_{jk}')_n=\dfrac{\sin(\psi_j')_n-(v_{jk}')_n\sin(\psi_k')_n}{\sin(\psi_j')_n+(v_{jk}')_n\sin(\psi_k')_n}\\[2ex](\Gamma_{jk}')_p=\dfrac{\sin(\psi_k')_p-(v_{jk}')_p\sin(\psi_j')_p}{\sin(\psi_k')_p+(v_{jk}')_p\sin(\psi_j')_p}\end{cases}$$
$$(5.25)$$

Substituting Equations 5.24 into Equations 4.34 and 5.25,

$$\Gamma_{ij}' = -\Gamma_{jk}',\ \text{for } v_k = v_i,\ d_i = d_k = 0 \quad (5.26)$$

The reflection coefficient $\Gamma'(1)$ of Equation 5.13 is independent of the number of lattice planes in medium j if and only if

$$\Gamma'(1)=\Gamma_{ij}',\begin{cases}\Gamma_{jk}'=0\\ \text{or}\\ (\Gamma_{ij}')^2=1\ (\Gamma_{ij}'=\pm 1)\end{cases}\qquad(5.27)$$

The condition $(\Gamma_{ij}')^2 = 1$ corresponds to the edges of the total reflection bands given by Equation 4.43*c* (total internal reflection) and by Equation 4.74*c* (Bragg reflection). More specifically, the condition $\Gamma_{ij}' = +1$ is identified with the angles $\psi_i = (\psi_{\bar{a},M})_{ij}$ and $\psi_i = (\psi_{D,N})_{ij} - (\Delta\psi_{D,N})_{ij}$ whereas the condition $\Gamma_{ij}' = -1$ is identified with the angles $\psi_i = (\psi_{\underline{a},M})_{ij}$ and $\psi_i = (\psi_{D,N})_{ij} + (\Delta\psi_{D,N})_{ij}$. The condition $\Gamma_{jk}' = 0$ corresponds to the angles of total transmission defined for the interface *jk*. More specifically, the condition $\Gamma_{jk}' = 0$ reduces to $\psi_j' = (\psi_{b,M})_{jk}$.

The general condition for the invisibility (zero reflection) of the waves incident on the bounded medium *j* of FIG. 18*c* is given by

$$\Gamma'(1)=0,\begin{cases}\Gamma_{ij}'/\Gamma_{jk}'=-(X_{ij})^{2m},\ \Gamma_{jk}'\neq 0\\ \Gamma_{ij}'=\pm\Gamma_{jk}',\ \Gamma_{jk}'=0\end{cases}\qquad(5.28)$$

If media *i*, *j*, and *k* have zero absorption loss, then for radiation at normal incidence, Equation 5.28*a* reduces to

$$\Gamma'(1)/_{\psi_1=\pi/2}=0,\begin{cases}v_i\neq v_k,\ v_j'=(v_i v_k')^{1/2},\ md_j=(\lambda_j'/4)(2k+1),\ k=0,1,2,3,\ \ldots\\ v_i=v_k,\ d_i=d_k=0,\ md_j=(\lambda_j'/2)(2k+1),\ k=0,1,2,3,\ \ldots\end{cases}\qquad(5.29)$$

Equation 5.28*b* reduces to

$$\Gamma'(1)=0,\begin{cases}v_i\neq v_k,\ \psi_i=(\psi_{b,M})_{ij},\psi_j'=(\psi_{b,M})_{jk}\\ v_i=v_k,\ d_i=d_k=0,\ \psi_i=(\psi_{b,M})_{ij}\end{cases}\qquad(5.30)$$

In Equations 5.29*b* and 5.30*b*, the conditions $v_i = v_k$, $d_i = d_k = 0$ are derived from Equation 5.26.

If media *i* and *k* are identical continuous media so that Conditions 5.26 are satisfied, then Equations 5.9 reduce to

$$\left.\begin{aligned}T_r/T_1&=(X_{ij})^{r-1}\left[\frac{1-(\Gamma_{ij}')^2(X_{ij})^{2(m-r+1)}}{1-(\Gamma_{ij}')^2(X_{ij})^{2m}}\right]\\ S_r/T_1&=\Gamma_{ij}'(X_{ij})^{r-1}\left[\frac{1-(X_{ij})^{2(m-r+1)}}{1-(\Gamma_{ij}')^2(X_{ij})^{2m}}\right]\end{aligned}\right\},\ v_k=v_i,\ d_i=d_k=0,\ 1\le r\le m+1\qquad(5.31)$$

Equations 5.31 are an improvement over the results obtained by Darwin[33], Mauguin[34], and Brillouin[35] for (x-rays incident on) a perfect crystal which is bounded by free space at both interfaces. Darwin[33] assumed the initial condition $S_{m+1} = 0$ (for the notation of FIG. 18*c*) and substituted this condition into the difference equations given by Equations 2.8. mauguin[34] demonstrated that the substitution of this initial condition into Equations 2.8 leads to solutions for S_r and T_r in terms of well-defined recursion polynomials. Brillouin[35] identified these polynomials as Gegenbauer functions of rank 1. With the use of Equation 2.14*a*, it can be shown that Mauguin's solution is equivalent to

$$T_r/T_1=(X_{ij})^{r-1}[1-\Phi_{ij}^2(\Gamma_{ij}')^2(X_{ij})^{2(m-r+1)}]/[1-\Phi_{ij}^2(\Gamma_{ij}')^2(X_{ij})^{2m}]$$
$$S_r/T_1=\Gamma_{ij}'(X_{ij})^{r-1}[1-(X_{ij})^{2(m-r+1)}]/[1=\Phi_{ij}^2(\Gamma_{ij}')^2(X_{ij})^{2m}]$$

where Φ_{ij} is given by Equation 5.18. For angles of incidence in the vicinity of the Bragg angles ($V_{ij} \approx N\pi$) and

for most electromagnetic wave-interactions ($V_{ij} << 1$), $\Phi_{ij}^2 \approx 1$. Irrespective of the value of Φ_{ij}, Equations 5.31 and 5.32 both give the results $S_{m+1}=0$ and $S_r/S_1 = (X_{ij})^{r-1}[1 - (X_{ij})^{2(m-r+1)}]/[1 - (X_{ij})^{2m}]$. Equations 5.32 differ from Equations 5.31 by the factor Φ_{ij}^2 because Darwin (and then Mauguin) incorrectly assumed that if $S_{m+1}=0$ at the $(m+1)$ th lattice plane, then the reflected wave S_r just above the $(m+1)$ th lattice plane is also zero where $m < r < m+1$. However, S_r is a standing wave in a bounded medium and the amplitude of the standing wave is non-zero at an infinitesimally small distance away from the null point at $r=m+1$. Therefore, in determining T_{m+1} and S_m, Darwin and Mauguin neglected the small contribution of the reflected wave S_r just above the $(m+1)$ th lattice plane. If S_r had been a travelling wave or if the initial condition had been non-zero, then the method of substituting the initial condition into the difference equations would have given correct results.

Consider Equations 5.31 for the case of a plane x-ray wavefront incident on a perfect crystal *j* which is bounded by free space at both interfaces. In particular, consider an angle of incidence $\psi_i = (\psi_{D,N})_{ij}$ corresponding to the center of the Darwin bands for Bragg reflection in medium *j*. For this angle of incidence, $\Gamma_{ij}' = i$ and $X_{ij} = \exp(i\,N\pi)\exp[-\bar{u}_{B,N}(1/Z_A)F_T(\psi_i,h)]$. Substituting these values into Equations 5.31, the reflection coefficient $\Gamma'(1)$ is given by

$$\Gamma'(1)/\Gamma_{ij}' = \tanh[m\,\bar{u}_{B,N}\,(1/Z_A)\,F_T(\psi_i,h)],\ \psi_i = (\psi_{D,N})_{ij},\ \text{Conditions 5.26}\qquad(5.33)$$

where

$$\bar{u}_{B,N} = (2\pi d_j/\lambda_i)(1/2\,\sin\psi_{B,N})(1-v_{ij}^2)=(1-v_{ij}^2)/2N\pi$$

For $m \to \infty$, $\Gamma'(1)/\Gamma_{ij}' \to 1$ and for $m \to 0$, $\Gamma'(1)/\Gamma_{ij}' \to 0$.

Equation 5.33 should be compared with the result

$$|\Gamma'(1)/\Gamma_{ij}'|^2 = [m\,\bar{u}_{B,N}(1/Z_A)\,F_T(\psi_i,h)]^{-1}\tanh[m\,\bar{u}_{B,N}\,(1/Z_A)\,F_T(\psi_i,h)]$$

obtained by Darwin[33] for x-rays from a point source (a spherical wavefront) incident at approximately the Bragg angle $\psi_{B,N}$.

6.0 TOWARDS A UNIFIED SCATTERING THEORY

In determining the interference of electromagnetic waves which are scattered by each scattering center of semi-unbounded and bounded media, this paper has attempted to resolve anomalies and to unify aspects of electromagnetic theory. Whereas Maxwell's equations, in their present form, are applicable to only the one coordinate (0,0) in $\bar{u},V$ phase space, the difference equations and single-scattering coefficients which have been developed in this paper are applicable to every coordinate in $\bar{u},V$ phase space. Since matter is discrete rather than continuous, Maxwell's equations are at best an idealized mathematical theory which is only approximately correct for those electromagnetic wave-interactions whose phase coordinates $(\bar{u},V)$ are in close proximity to the origin (0,0). In contrast, the theory of this paper is exact (within the limitations that have been imposed) at every coordinate $(\bar{u},V)$ and is intimately related to the physical phenomena associated with each scattering event in phase-ordered media.

Several conditions have been imposed upon the results. Some methods and problems associated with the removal of these conditions are now discussed.

1. The concept of phase interference is meaningful in terms of steady-state monochromatic waves. The concept can also be applied to other periodic waveforms and to transient waveforms if the input waveform is expressed in terms of its Fourier or Laplace frequency components. Standard convolution techniques can therefore be applied to determine the reflected and transmitted waveforms for input waveforms of arbitrary shape.

2. The radiation scattered by each electron of the medium has been assumed to have a frequency identical to that of the incident radiation, a phase directly related to the phase of the incident radiation, and a directional amplitude of complete certainty characterized by Rayleigh scattering. These assumptions are valid only if there are no pertubations or changes in the energy state of the electrons. Brillouin scattering[9] and the Compton effect[36,37] are examples of incoherently scattered radiation resulting from changes in the energy state of the electrons. In the case of Brillouin scattering, the direction and frequency of the scattered radiation is a function of the acoustical vibrations of uncertain amplitude caused by the thermal energy of the lattice or atoms. In the case of the Compton effect, the direction and frequency of the scattered radiation is a function of the electron recoil and its uncertainty to an elastic collision with a high energy (hard x-ray) photon of the incident radiation. In these examples of incoherently scattered radiation, only a portion of the incident wavefront is scattered incoherently. The resultant reflected and forward waves are partially coherent, partially monochromatic, and partially specular. Coherent atomic structure factors may be modified and incoherent atomic structure functions may be introduced, as is done in x-ray physics[16], to account for changes in the energy state of electrons. However, until the uncertainties associated with these energy changes are resolved, the interference of waves scattered from such electrons must also remain uncertain.

3. The parameters δV_{rms} and $\delta\bar{u}_{rms}$, which characterize the phase-disorder of media, have been assumed to satisfy Conditions 2.7 if the difference equations are to be applicable. Perfect phase order occurs only in continuous media or in perfect crystals whose atoms are arranged in equally spaced planes parallel to the interface. In such media, $\delta V_{rms} = \delta\bar{u}_{rms} = 0$ and the interference of scattered waves results in specularly reflected and transmitted waves. In this paper, no attempt has been made (except for the case $\delta V_{rms} = \delta\bar{u}_{rms} = 0$) to establish the functional relationship of the phase-disorder parameters to the amplitudes of the scattered waves.

4. The permeability of the media has been assumed to be identical to that of free space. Only the harmonic linear displacement, of electrons excited by the incident electric field intensity, has been considered. If the theory of this paper is to be extended to magnetic media and is still to be derived independently of Maxwell's equations, then it will be necessary to consider also the harmonic changes in the spin and orbital motion of the electrons comprising the media.

5. The scattering cross-section of each electron has been assumed to be sufficiently small (see Equation 3.7) so that each scattering center in a given lattice plane scatters the incident wavefront independently of one another. This assumption is valid for electromagnetic waves with the possible exceptions of grazing angles of incidence almost identical to zero and wavelengths in the immediate vicinity of an absorption line. The interference of waves from coupled-scattering centers has been studied for low-energy electron waves[38] and for the general class of waves in periodic structures[39]. In the former case, the diffraction from parallel lattice planes is combined with a method of self-consistent fields. In the latter case, a more general difference wave equation than Equation 2.9 is employed to account for mutual forces between scattering centers. Both of these cases develop useful methodologies but are incomplete because they fail to correctly relate wave refraction to the phase difference $\bar{u}$ of the forward scattered wavefront. As a consequence, in the latter case for example, the difference wave equation is abandoned for the differential wave equation and the pertubation theory (or selected periodic functions) associated with it[40].

6. Any discrete medium, whose phase disorder does not violate Conditions 2.7, can be approximated to at least a first order approximation by scattering centers which are arranged in equally spaced planes, of spacing d_{av}, parallel to the interface of the incident wavefront. For example, if $V_{av} << 1$ and $|\bar{u}_{av}| << 1$, then the theory of this paper is applicable to any crystal or amorphous medium. However, if either $V_{av} \gtrsim 1$ or $|\bar{u}_{av}| \gtrsim$ 24 1, then Conditions 2.7 are satisfied only by those crystalline lattice structures in which the atoms are arranged in equally spaced planes parallel to the interface. For these crystalline lattice structures of perfect periodicity which do not satisfy Conditions 2.7, the atoms are also arranged in planes parallel to the interface. However, in this case, the spacings between planes are not all equal but instead have more than one periodicity. The problem therefore reduces to the theory of difference equations with more than one periodicity. An alternative approach is to define a unit cell as the diffracting unit rather than a lattice plane. The unit cell is defined by Ewald's reciprocal lattice parameters[8,41]. The interference of waves, with the unit cell as the diffracting unit, has never been determined exactly. However, approximate solutions for Bragg-type reflection in three-dimensional lattices have been obtained by assuming that the effective wavelength at the center of the reflection bands is either equal to the incident wavelength[42] or is equal to the incident wavelength divided by the relative wave velocity v_{ij} for continuous media[40]. The correct effective relative wave velocity v_{ij}' in the vicinity of a Bragg angle (for equal lattice spacings) is given by Equation 4.80 which is derived from difference equations rather than differential equations. Brillouin[43] has determined the geometric zones (using Ewald's construction) associated with Bragg-type reflection for several three-dimensional crystalline structures. Brillouin zones are identified with the phase pa-

rameter V since Bragg reflection occurs for $V \approx N\pi$. In addition to Brillouin zones, it is possible to determine zones which are identified with the phase parameter $\overline{u}$ since total reflection also occurs for $\overline{u} \approx (2M+1)(\pi/2)$.

In the preceding discussion of Conditions 5 and 6, it was suggested that waves in discrete media cannot be correctly treated by a differential wave equation unless the effective refractive parameters are substituted into it. This point is illustrated by the following comparison of differential and difference wave equations. The differential wave equation for the projection onto the z axis of a wavefront propagating in the continuous medium j of FIG. **10**b is given by

$$(\sigma^2\tau/\sigma z^2) + (2\pi/\lambda_{jz})^2\tau = 0 \tag{6.1}$$

where τ = transmitted electric field intensity at a coordinate z on the z axis of medium j. $\lambda_{jz} = \lambda_{iz}/v_{ij}$ = wavelength in the direction of the z axis in a continuous medium j. However, as was pointed out in Section 5.0, the transmitted electric field intensity in any phase-ordered medium, whether it be discrete or continuous, may be expressed as the sum of a forward wave T and a reflected wave S whose phases propagate in the same direction and at the same rate. From Equations 5.11, the transmitted electric field intensity τ in a continuous medium j is given by

$$\vec{\tau} = \vec{T} + \vec{S} = \begin{cases} \tau_n = T_n[1+(\Gamma_{ij})_n] \\ \tau_p = [T_p(\sin\psi_i)/\sin\psi_j][1+(\Gamma_{ij})_p] \end{cases} \tag{6.2}$$

Therefore Equation 6.1 may be expressed as two differential wave equations given by

$$(\sigma^2\, T/\sigma z^2) + (2\pi/\lambda_{jz})^2\, T = 0,\ (\sigma^2\, S/\, \sigma z^2) + (2\pi/\lambda_{jz})^2 S = 0 \tag{6.3}$$

The corresponding difference equation for the forward wave T_r (and an identical one for the reflected wave S_r) in the discrete medium j of FIG. **10**a is given by Equation 2.9. Equation 2.9 can be expressed in a form which more closely resembles that of Equation 6.3. Dividing by d^2 (where d is the lattice spacing) and rearranging terms, Equation 2.9 may be written as

$(1/d^2)(T_{r+1} - 2T_r + T_{r-1}) + (2/d^2)(1-A)T_r = 0$ where A is given by Equation 2.10. If we now introduce the central difference operator δ defined by[13]

$\delta T_r = T_{r+1/2} - T_{r-1/2}$, then $\delta^2 T_r = \delta(\delta T_r) = T_{r+1} - 2\, T_r + T_{r-1}$. The same results are obtained for the reflected wave S_r. Therefore, the difference wave equations for the forward and reflected waves reduce to

$$[(\delta^2 T_r)/d^2] + (2/d^2)(1-A)T_r = 0,\ [(\delta^2 S_r)/d^2] + (2/d^2)(1-A)S_r = 0 \tag{6.4}$$

It is important to recognize that the quantity $(2/d^2)(1-A)$ is generally not equal to $(2\pi/\lambda_{jz}')^2$ where λ_{jz}' is the effective wavelength in the direction of the z axis in a discrete medium j. From Equations 2.11, 2.15, and 2.17, λ_{jz}' is given by

$$(2\pi/\lambda_{jz}')^2 = -(1/d)^2(\ln X)^2 = (1/d)^2(\cos^{-1}A)^2 \tag{6.5}$$

If the single-scattering coefficients of Equations 2.2, 3.29, and 3.30 are substituted into Equation 2.10, then parameter A exhibits the properties

$$A: \begin{cases} A \neq 1, (d \neq 0, \text{ Conditions 6.6}c \text{ not satisfied}) \\ \lim A \to 1 \text{ as } d \to 0 \\ A = 1, (V = 2N\pi, \overline{u} = M\pi, (1/Z_A)F_T(\psi,h) = 1 \end{cases} \tag{6.6}$$

It then follows that

$$(2/d^2)(1-A) \begin{cases} \neq (2\pi/\lambda_{jz}')^2,\ (d \neq 0, \text{ Conditions 6.6}c \text{ not satisfied}) \\ = (2\pi/\lambda_{jz})^2,\ d \to 0 \\ = (2\pi/\lambda_{jz}')^2,\ (V = 2N\pi,\ \overline{u} = M\pi,\ (1/Z_A)F_T(\psi,h) = 1 \end{cases} \tag{6.7}$$

From Equation 6.7b, the difference wave Equations 6.4 reduce to the differential wave Equations 6.3 in the limiting case of continuous media. From Equation 6.7a, for discrete media, Equations 6.4 are not strictly analogous to Equations 6.3 with the exception of the case given by Equation 6.7c. Whereas λ_{jz}' is derivable from the solution of Equations 6.4, attempts to derive λ_{jz}' from the solution of Equations 6.3 have either been unsuccessful or have been only approximately correct in certain restricted cases[29,40]. If λ_{jz}' is determined a priori (for example, by solution of Equations 6.4) and is substituted for λ_{jz} in Equations 6.3, then Equations 6.3 will give the same results as the solution of Equations 6.4 for discrete media.

Differential wave equations of the same form as Equations 6.3 are applicable to any kind of plane wave (electromagnetic, elastic, or wave-mechanical De Broglie wave) in continuous media. Similarly, difference wave equations of the same form as Equations 6.4 are applicable to any kind of plane wave in phase-ordered media (continuous or discrete). Since matter is discrete rather than continuous, difference wave equations, which relate multiple-scattering wave amplitudes to single-scattering events, give more unified and exact results than is obtainable from differential wave equations. For this reason, theoretical physics should more logically be dominated by difference equations which are based on single-scattering concepts rather than by differential equations which are based on field concepts. Difference wave equations of the form of Equations 6.4 are self-consistent because they are always derivable from two self-consistent first order difference equations of the form given by Equations 2.8. Equations 2.8 are self-consistent because the wave amplitudes T_r and S_r have identical physical dimensions of some physical characteristic of the wave. For example, T_r and S_r may be an electric field intensity for an electromagnetic wave, a linear displacement for an elastic wave, or a flux density for a wave-mechanical De Broglie wave. The self-consistent nature of the difference equations and the extremely narrow transmission and reflection bandwidths that can result from these equations suggest an alternative approach to quantum mechanics which is presently treated as a self-consistent probability eigenvalue problem in the form of a differential wave equation. The results of this paper are exactly applicable to any type of plane wave in phase-ordered media which satisfys the conditions that have been imposed upon the results. For example, it is expected that the results of this paper will find application in calculating the detailed structure of the reflection and transmission bands associated with acoustic and electron waves in phase-ordered media. The results should not only provide a more exact solution for Bragg reflection bands ($V \approx N\pi$) but should also lead to the discovery of new transmission and reflection bands for the conditions $\bar{u} \approx M\pi$ and $\bar{u} \approx (2M+1)(\pi/2)$ respectively.

The theory, of waves in phase-ordered media, applies not only to atomic lattices but also to artificial lattices. Periodic loaded transmission lines and media (in which lumped obstacles are imbedded in a dielectric) are some examples of artificial lattices to which this theory can be applied. A further example is an elastic wave which periodically modulates a medium through which an electromagnetic or electron wave propagates. The wavelengths associated with particular reflection or transmission bands may be orders of magnitude larger in artificial lattices than in atomic lattices because the lattice spacing d can be made correspondingly larger.

The theory of waves in phase-ordered media is a theory of multiple scattering based on single-scattering concepts. Table 8 compares some characteristics of single-scattered waves with those of multiple-scattered waves in phase-ordered media. Prior to this paper it had not been possible, with the exception of x-ray Bragg reflection[1], to relate multiple scattering in phase-ordered media to single-scattering concepts.

Multiple scattering is usually associated with problems of radiative transfer and transport phenomena. In terms of the phase-disorder parameters δV_{rms} and $\delta \bar{u}_{rms}$, radiative transfer may be defined as the propagation of waves in media of any phase-disorder ($0 \leq \delta V_{rms} < \infty$, $-\infty \leq \delta \bar{u}_{rms} < \infty$) whereas transport phenomena (such as diffusion, heat conduction, viscosity) may be defined as the propagation of waves in media of large phase-disorder ($\delta V_{rms} \gtrsim \pi$ or $|\delta u_{rms}| \gtrsim \pi$). A medium of large phase-disorder should not be confused with an amorphous medium since it has already been shown that an amorphous medium may satisfy Conditions 2.7. Since this paper determines the radiative transfer in those media which satisfy Conditions 2.7, the results are related to the solution of the general class of problems associated with radiative transfer including transport phenomena.

TABLE 8.—CHARACTERISTICS OF SINGLE-SCATTERED AND MULTIPLE-SCATTERED WAVES IN PHASE-ORDERED MEDIA

Parameter	Property	Symbol or condition	
		Single-scattered wave	Multiple-scattered wave
Phase difference between lattice planes	In forward direction	$\bar{u}$	$\left(\frac{2\pi d}{\lambda}\right)\left(\frac{1}{\sin\psi_i} - \frac{v_{ij}'}{\sin\psi_j'}\right)$.
	In reflected direction	V	$(2\pi d/_2)\ v_{ij}' \sin\psi_i$.
Coefficient	Reflection	a	S_1/T_1.
	Transmission	t	τ_1/T_1.
	Amplitude delay	Φ	X.
Band	Total transmission	$\bar{u}=M\pi$ or $V=(2N+1)(\pi/2)$	$U=M\pi$.
	Total reflection	$\bar{u}=(2M+1)(\pi/2)$ or $V=N\pi$	$\frac{\cos^2(V-u)}{\cos^2 U} \geq 1$.

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What is claimed is:

1. Apparatus for selectively propagating waves in a direction having a component parallel to an interface comprising,

means defining discrete phase-ordered media along said interface characterized by a wave refraction constant and a spacing d between lattice planes generally parallel to said interface,

a source of said waves,

and means for establishing the wavelength, polarization and angle of incidence upon said interface of said waves to selectively propagate said waves in a direction having a component parallel to said interface in a pre-determined manner controlled by said wave refraction constant and said spacing d and establish a substantial phase difference between wave fronts at adjacent ones of said lattice planes in the forward scattered direction at least of the order of a radian.

2. Apparatus in accordance with claim 1 wherein said waves are electromagnetic and metal comprises said media.

3. Apparatus in accordance with claim 1 wherein said waves are electromagnetic and dielectric comprises said media.

4. Apparatus in accordance with claim 3 wherein said angle of incidence is a small grazing angle slightly greater than zero degrees with said wave refraction constant and said lattice spacing being such that only narrow bands of electromagnetic waves of predetermined polarization and wavelength propagate in said direction.

5. Apparatus in accordance with claim 4 and further comprising means for adjusting and measuring said angle of incidence,

and means for detecting the wavelength, intensity and polarization of the waves propagated by said apparatus in said direction.

6. Apparatus in accordance with claim 4 wherein at least one small angle of incidence produces substantially total reflection.

7. Apparatus in accordance with claim 5 wherein said means for detecting includes means for detecting transmitted waves and means for detecting reflected waves,

and further comprising means for stressing said media to effect the ratio of transmitted wave intensity to reflected wave intensity.

8. Apparatus in accordance with claim 6 wherein said media surround different media of wave refraction constant less than that of said first-mentioned media whereby waves may propagate through said second media by multiple reflections from said first-mentioned media.

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