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INTEGRATING APPARATUS

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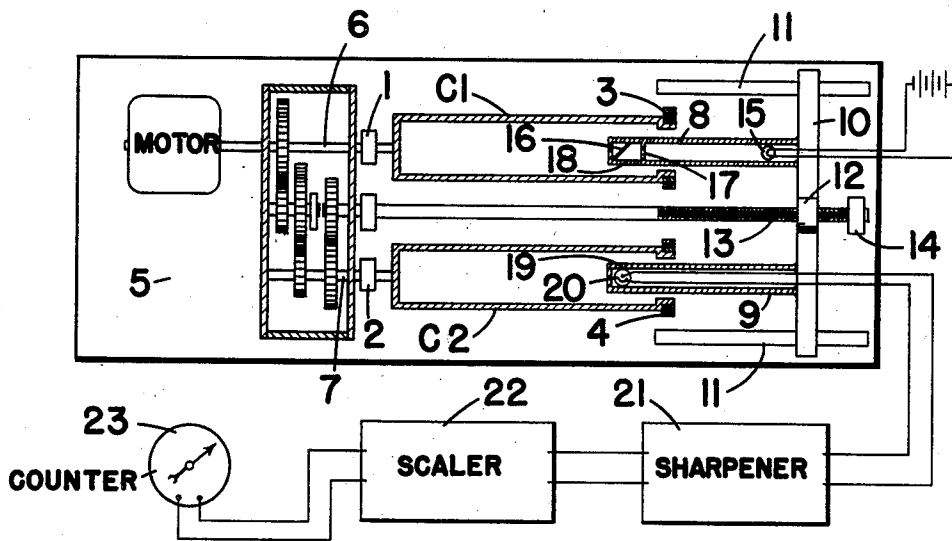


FIG. 1.

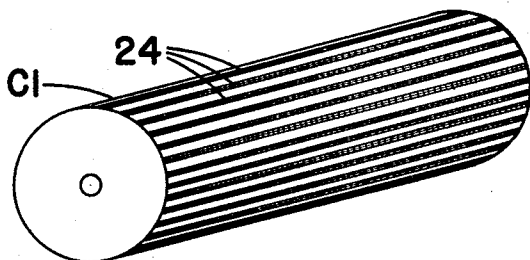


FIG. 2.

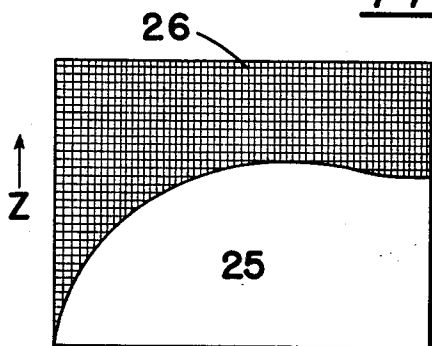


FIG. 3.

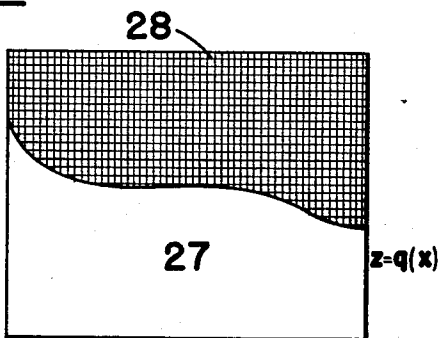


FIG. 4.

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INTEGRATING APPARATUS

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4 Claims. (Cl. 235—61)

The present invention is directed to a method and apparatus for integrating mathematical functions.

In many physical and mathematical problems it is necessary to perform repeatedly integra-
tions of the form

$$f(t) = \int_0^t p(x-y)q(x)dx$$

wherein $p(x)$ and $q(x)$ may be known only as curves and $y=0$ or a constant. Such integrations are commonly carried out arithmetically by plotting the two curves, measuring the value of each function at equally-spaced values of x , multiplying the values of $p(x-y)$ and $q(x)$ for each measured value of x , and adding the resulting products. Since this process is laborious and time consuming, various machines have been devised for performing this operation mechanically or electrically. A representative type of such machines is described in "The Cinema Integrator," Hazen, Brown and Hedeman, Journal of the Franklin Institute, July 1940, in which may also be found illustrations of the type of integrals, to the solution of which the present invention is directed.

The principal object of the present invention is the provision of a relatively simple method and apparatus for performing integrations of the character referred to with any desired degree of accuracy. Briefly, the method involves the creation of light flashes, the number of which is a function of the curves represented by $p(x-y)$ and $q(x)$ and, more specifically, a function of the product of these two curves.

The present invention may be more clearly understood from the following detailed description of the accompanying drawing in which

Fig. 1 is a plan view of one embodiment of the present invention;

Fig. 2 is a detail thereof;

Fig. 3 is a detailed view of a light mask used in the arrangement shown in Fig. 1; and

Fig. 4 is a similar view of a different light mask.

Briefly, the arrangement shown in Fig. 1 consists of a pair of revolving masks interposed between a light source and a photocell or other device capable of registering light flashes. One of the masks has drawn on it a curve represented by the equation $z=p(x)$ and the other has drawn on it a curved represented by the equation $z=q(x)$. Both masks have the x axis parallel to the axis of rotation and their axes of rotation are parallel to each other and perpendicular to the

beam of light. Each mask is so constructed that for each revolution it will pass a number of flashes of light which will be a function of the value of z for the value of x at the point on the mask where the light beam strikes. The R. P. M. of one mask is made a large multiple of the R. P. M. of the other mask, and upon the size of this multiple will partially depend the accuracy of the integrations.

Referring to Fig. 1 in detail, C_1 and C_2 designate two hollow, transparent cylinders supported by bearings 1, 2, 3 and 4. These bearings are the conventional ring type ball bearings. It will be observed that the driving shafts 6 and 7, respectively, for these cylinders are journaled in bearings 1 and 2, while the outer surface of the cylinders themselves are journaled in bearings 3 and 4. This is necessary because the outer ends of the cylinders are open. The bearings are suitable mounted on a base plate 5. Shaft 6 is driven by a suitable motor, and is geared to shaft 7 in the manner shown. It will be apparent that any gear ratio may be employed in this transmission.

In the open ends of the cylinders, tubes 8 and 9 are mounted so as to have their axes coincident with the axes of rotation of cylinders C_2 and C_1 , respectively. These tubes are rigidly mounted in a block 10 which is arranged to slide on spaced tracks 11 attached to the base plate 5. The block 10 carries a screw threaded rider 12 which is in threaded engagement with a screw 13, one end of which is journaled in a bearing 14 and the other end of which terminates in a gear in the gear box in mesh with a driving gear fixed to the shaft 7. The pitch of the screw 13 can be arbitrarily selected, but the smaller it is made the greater will be the accuracy of the integration.

Fixed in the tube 8 is a suitable light source 15, such as an automobile lamp bulb, and a mirror 16 adapted to receive a beam of light from source 15 through a pin hole 17 and reflect it through a pin hole 18 in the wall of tube 8 in a line perpendicular with the axes of rotation of cylinders C_1 and C_2 and in the plane thereof. Tube 9 is provided with a hole 19 to pass said beam of light. Fixed in tube 9 in register with said hole is a light sensitive device 20, such as a photocell, which in the embodiment shown is connected electrically to an amplifying and sharpening circuit 21. A suitable sharpening circuit for this purpose is that described by Huntton and Strohmeier in an article in the Review of Scientific Instruments, 12, 35 (1941) entitled A hard vac-

uum tube pulse equalizing sharpening circuit. The purpose of this circuit is to convert the pulses of electricity generated by the photocell 20 into a suitable form to operate a scaling circuit 22 to which they are delivered. Connected to the scaling circuit is a counter 23 which displays a numerical summation of the pulses delivered to it.

As shown in Fig. 2, the surface of the transparent cylinder C_1 carries a series of opaque bands 24 parallel to each other and to the axis of the cylinder. In general these bands will be very narrow and so spaced that the transparent and opaque strips are of equal width. A preferred method of producing the stripes is to form them on transparent material, such as Cellophane, with India ink and then fasten the transparent sheet on the cylinder C_1 by means of glue of transparent type. It will be observed that only cylinder C_1 is so striped.

In Figs. 3 and 4 are shown masks which may be mounted on cylinders C_1 and C_2 . For the practice of the invention it is immaterial which mask is mounted on which cylinder. In Fig. 3, for example, the mask has a transparent portion 25, the upper boundary of which follows a curve designated by the equation $z=p(x)$ in which z represents the ordinate and x represents the abscissa. The remaining portion 26 of the mask is opaque. This mask is wound on the cylinder with its x axis parallel to the axis of rotation of the cylinder.

In Fig. 4 the mask illustrated has a transparent portion 27. The upper boundary is a curve represented by the equation $z=q(x)$. The remaining portion 28 of this mask is opaque. It will be obvious that the portions 25 and 27 of the respective masks can be cut out entirely. The masks are preferably made by tracing the curves on transparent sheets and then blackening the portions 26 and 28.

The masks are of such dimensions that the maximum value of z is equal to the circumference of the corresponding cylinder, and the maximum value of x is not greater than the length of the corresponding cylinder. As has been previously indicated, either mask can be placed on either cylinder, but for the purpose of discussion it will be assumed that the mask of Fig. 3 is placed on cylinder C_1 and the mask of Fig. 4 is placed on cylinder C_2 .

It should now be clear that, as the cylinders rotate, the beam of light passing through C_1 and C_2 will be chopped into flashes by the stripes on C_1 and that this flashing beam will be interrupted at intervals both by C_1 and C_2 as the opaque portions of the masks intercept the beam. The resultant pulsating signal from the photocell is amplified and the pulses sharpened as previously described by circuit 21 and counted.

To operate the device the masks are placed on the cylinders and the tubes 8 and 9 are positioned so that the beam of light passes through $x=-y$ and $z=0$ on the mask on C_1 , and $x=z=0$ on the mask on C_2 . Then, with the amplifier and scaler ready to function, the reading of the electric counter is noted or is set at zero. Then the motor is started and allowed to run until the screw 13 has advanced the beam of light to the point corresponding to $x=t-y$ on the mask on cylinder C_1 and $x=t$ on the mask on C_2 where t is the upper limit of the integral. At this point the electrical counter is disconnected from the scaler, the motor is turned off, and the reading of the counter is noted.

The manner in which the machine performs

the desired multiplication and summation will now be considered. Assume that the apparatus has been started with the ray of light passing through the point $x=-y$ and $z=0$ on C_1 and $x=0=z$ on C_2 . Let the total number of clear strips on C_1 be N (disregarding the mask) and let the circumference of C_1 be D_1 and the circumference of C_2 be D_2 . Let the gear ratios be such that C_1 rotates n revolutions to one revolution of C_2 , and let the pitch of screw 13 be such that for one revolution of C_2 the beam of light advances a distance $S=c\Delta x$ in the positive x direction, where Δx is the increment in x corresponding to S and c is the scale; that is, c is the ratio of the length of a given increment in the x direction to its value in the units in which x is expressed. For purposes of clarity, it will be assumed in the following discussion that $p(x-y)$ and $q(x)$ remain constant over any given increment, Δx . The maximum error introduced by this approximation will clearly be proportional to Δx .

During the first revolution of C_1 the number of flashes of light, μ_1 , passing through C_1 is proportional to

$$\frac{N}{D_1} p(-y)$$

or

$$\mu_1 = a \cdot \frac{N}{D_1} p(-y)$$

where a is the constant ratio of the height of $p(x-y)$ as measured on C_1 from the x axis to the numerical value of $p(x-y)$. In other words, it is a constant which is a characteristic of the mask itself, and represents the scale to which the curve on the mask is plotted.

During the different revolutions of C_1 , C_2 rotates D_2/n times in the positive z direction and the light beam advances a distance S/n in the positive x direction.

Assuming that b times $q(0)$ is greater than D_2/n , where b is a constant for C_2 corresponding to a for C_1 , μ_1 flashes of light will also pass through C_2 and actuate the photocell and be registered by the scaler and counter. Furthermore, μ_1 flashes of light will continue to be registered per revolution of C_1 until C_2 has rotated through the distance b times $q(0)$. The number of revolutions, m_1 , of C_1 corresponding to this fraction of a revolution of C_2 is the number of revolutions of C_1 per revolution of C_2 times the ratio of b times $q(x_1)$ to the total circumference of C_2 . That is

$$m_1 = n \cdot b \cdot \frac{q(0)}{D_2}$$

During the remaining part of the revolution of C_2 , the mask thereon prevents flashes of light from reaching the photocell. Therefore, the total number of flashes of light impinging on the photocell during the first revolution of C_2 is $F_1 = \mu_1 m_1$ or

$$F_1 = a N p(-y) \cdot \frac{n b q(0)}{D_1 D_2} = \frac{a b c n N}{D_1 D_2 S} p(-y) q(0) \Delta x$$

where numerator and denominator have been multiplied by $S=c\Delta x$. Since during subsequent revolutions of C_2 the above process will be repeated, we may write

$$F_i = K p(x_i - y) q(x_i) \Delta x$$

where the constants have been lumped into K , and where the subscript refers to the i th revolution of C_2 . The grand total number of flashes

of light registered as the light beam advances from $x=0$ to $x=t$ will then be

$$\sum_{i=1}^r F_i = K \sum_{i=1}^r p(x_i - y) q(x_i) \Delta x$$

where $r=ct/S$. But if Δx is small, this summation approaches the value of the integral, so that we have

$$\sum_{i=1}^r F_i \doteq K \int_0^t p(x-y) q(x) dx$$

or

$$\int_0^t p(x-y) q(x) dx \doteq \frac{1}{K} \sum F_i = \frac{d}{K} R$$

where R is the difference between the final and initial counter readings and d is the scaling factor. The error in the results will evidently be at least roughly proportional to $\Delta x/nN$, and it is consequently desirable to make this quantity as small as practicable. It is for this reason that a scaling circuit is used, for by this means a much larger value of N can be used than would otherwise be possible to register and still operate the machine at a reasonably fast rate. Examples of practical values of these constants for moderate accuracy are $\Delta x=1$ mm., $n=N=100$.

In the foregoing discussion, reference has been made to an equation containing the values $p(x-y)$ and $q(x)$. These values have been represented on the masks shown in Figs. 3 and 4 as being curves of a certain shape. It will be obvious that the curve may assume any shape, depending upon the specific functions of x represented by these values.

Other methods for generating pulses of light of a number which will be a function of the product of the values of $p(x-y)$ and $q(x)$ and will represent the summation of these products for different values of x between zero and t than that specifically disclosed will occur to those skilled in the art. One obvious modification is to omit the stripes from cylinder C_1 and generate the initial light flashes by inserting a shutter in the form of a toothed wheel between the light source 15 and the mirror 16 in the manner disclosed in my copending application Serial No. 405,910. In this case, both the shutter and the light source will be arranged outside the tube 8 so that means for driving the shutter at a speed which bears a fixed ratio to the R. P. M.'s of the cylinders C_1 and C_2 can be utilized.

As has been pointed out in the foregoing description, the accuracy of the device can be increased by increasing the gear ratio between C_1 and C_2 , decreasing the width of the stripes on C_1 , and decreasing the pitch of the thread on screw 13. It can be seen, therefore, that the machine is capable of considerable adjustment, depending upon the type of problem to be dealt with and the degree of accuracy desired.

In the above description it has been assumed that $p(x-y)$ and $q(x)$ are both always positive. In the event that this is not true, separate masks may be made for the positive and negative portions of the two functions. The integral will then be made up of four parts:

(a) the positive products corresponding to the positive portions of each function;

(b) the negative products of the negative por-

tion of the first times the positive portion of the second;

(c) the negative products of the positive portion of the first times the negative portion of the second; and

(d) the positive products corresponding to the negative portion of the first times the negative portion of the second.

It is clear that these separate parts of the integral may be obtained by four separate integrations, and the total integral is then determined by the algebraic sum.

The nature and objects of the present invention having been thus described and illustrated, what is claimed as new and useful and is desired to be secured by Letters Patent is:

1. An apparatus for integrating an equation of the type

$$f(t) = \int_0^t p(x-y) q(x) dx$$

where y is 0 or a constant and $p(x)$ and $q(x)$ can be represented by curves, comprising a light source, a light receiver spaced from said source and in alignment therewith, a plurality of rotative screens interposed between said light source and said receiver, markings on said screens to cause the light of said source to be transmitted to said receiver in a number of flashes, the number of which represents the integrated product of $p(x)$ and $q(x)$, and means for counting said flashes of light.

2. An apparatus according to claim 1 in which each of a pair of said screens has an opaque portion and a transparent portion, the transparent portion of one being an area defined by the curve represented by the equation $z=p(x)$, and the transparent portion of the other being an area defined by the curve represented by the equation $z=q(x)$.

3. An apparatus for integrating an equation of the type

$$f(t) = \int_0^t p(x-y) q(x) dx$$

where y is 0 or a constant and $p(x)$ and $q(x)$ can be represented by curves comprising a cylinder having a transparent portion and an opaque portion, a light source arranged along the axis of said cylinder adapted to cast a beam of light normal to the wall thereof, a second cylinder arranged with its axis parallel to the axis of the first cylinder and having an opaque portion and a transparent portion, the transparent portion of one of said cylinders being an area defined by the curve $z=p(x)$ and the transparent portion of the other being an area defined by the curve $z=q(x)$, the x axis in each case being parallel with the axis of the cylinder, a photocell arranged on the axis of said second cylinder in a position to receive a light beam from said source of light, means for rotating both cylinders while maintaining a selected gear ratio between them, means for chopping said light into a series of flashes whereby said photocell generates periodically pulses of current, and means for counting said pulses of current.

4. An apparatus according to claim 3 in which the means for chopping the light beam into flashes constitutes spaced opaque longitudinal stripes on one of said cylinders.

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