(19) **United States**

(12) **Patent Application Publication** (10) Pub. No.: **US 2003/0055630 A1**

Byrnes et al. (43) Pub. Date: **Mar. 20, 2003**

(54) **METHOD AND APPARATUS FOR A TUNABLE HIGH-RESOLUTION SPECTRAL ESTIMATOR**

(75) Inventors: **Christopher I. Byrnes**, St. Louis, MO (US); **Anders Lindquist**, Taby (SE); **Tryphon T. Georgiou**, Falcon Heights, MN (US)

Correspondence Address:
THOMPSON COBURN, LLP
ONE FIRSTAR PLAZA
SUITE 3500
ST LOUIS, MO 63101 (US)

(73) Assignee: **Washington University**

(21) Appl. No.: **10/162,182**

(22) Filed: **Jun. 4, 2002**

Related U.S. Application Data

(62) Division of application No. 09/176,984, filed on Oct. 22, 1998, now Pat. No. 6,400,310.

Publication Classification

(51) Int. Cl.^{7} G10L 19/14

(52) U.S. Cl. 704/205

(57) **ABSTRACT**

A tunable high resolution spectral estimator is disclosed as a method and apparatus for encoding and decoding signals, signal analysis and synthesis, and for performing high resolution spectral estimation. The invention is comprised of an encoder coupled with either or both of a signal synthesizer and a spectral analyzer. The encoder processes a frame of a time-based input signal by passing it through a bank of lower order filters and estimating a plurality of lower order covariances from which a plurality of filter parameters may be determined. Coupled to the encoder, through any appropriate data link or interface including telecommunication links, is one or both of a signal synthesizer and a spectral analyzer. The signal synthesizer includes a decocer for processing the covariances and a parameter transformer. The signal synthesizer includes a decoder for processing the covariances and a parameter transformer for determining filter parameters for an ARMA filter. An excitation signal is processed through the ARMA filter to reproduce, or synthesize, a representation of the input filter. The spectral analyzer also includes a decoder which processes the covariances for input to a spectral plotter to detemine the power frequency spectrum of the input signal. The invention may be used in a myriad of applications including voice identification, doppler-based radar speed estimation, time delay estimation, and others.
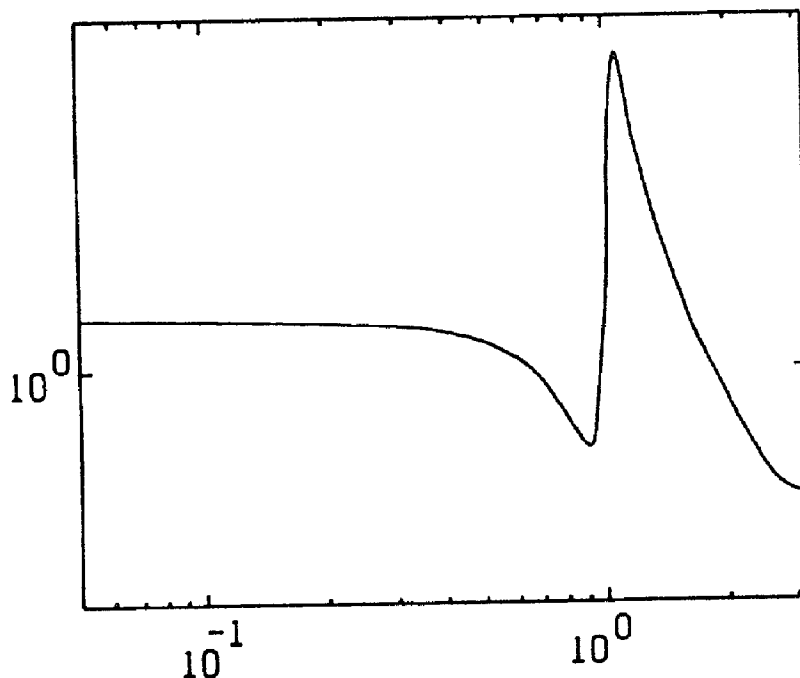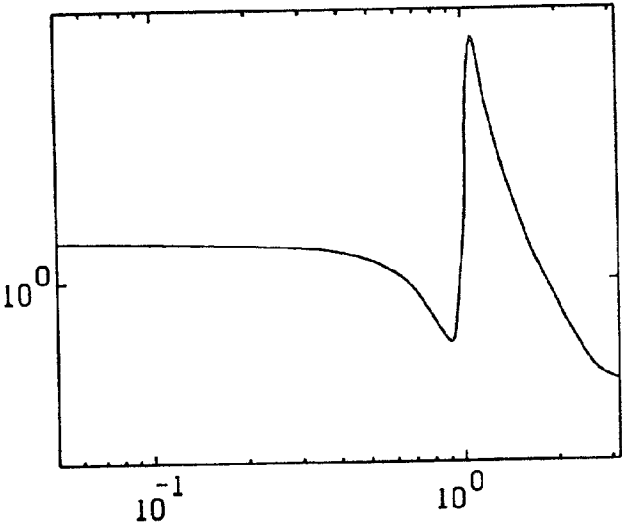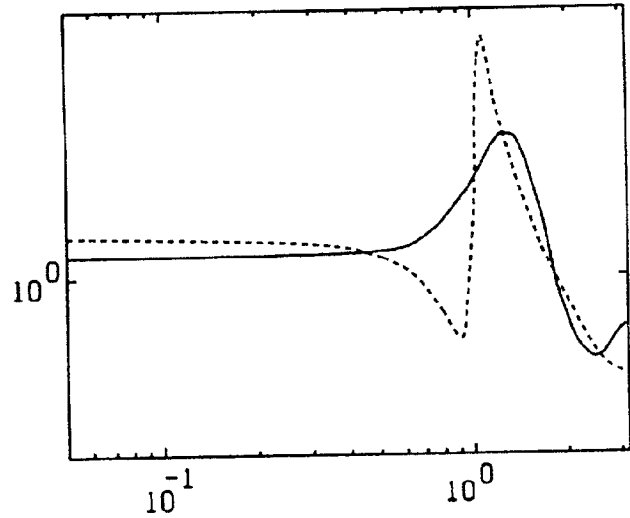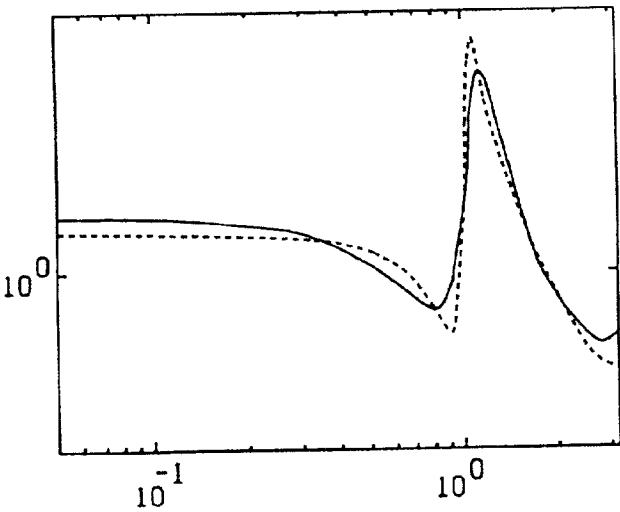
FIG.1

FIG.2

FIG.3
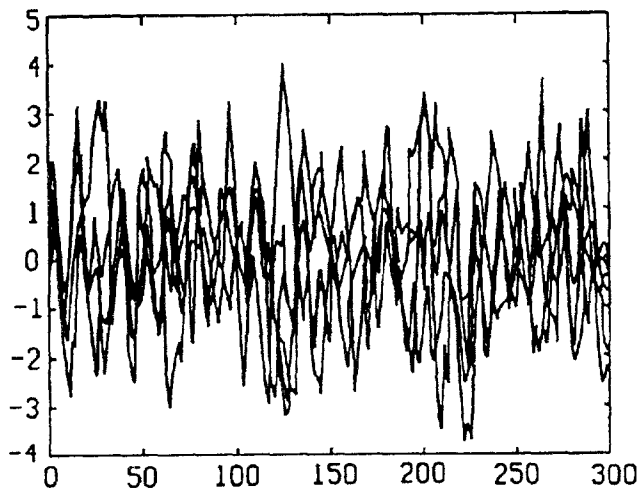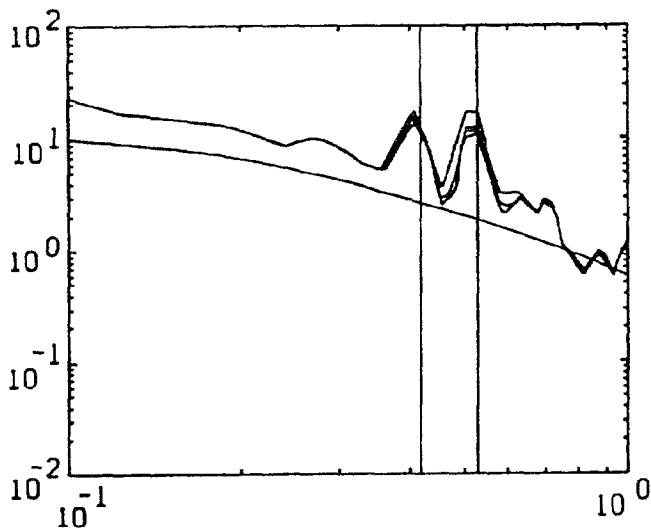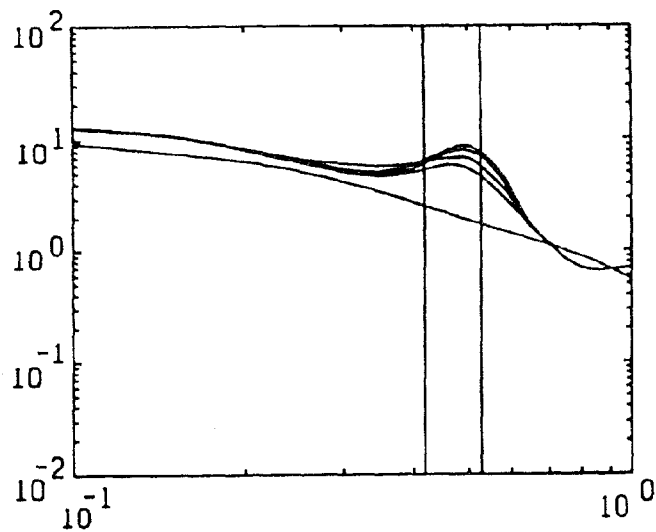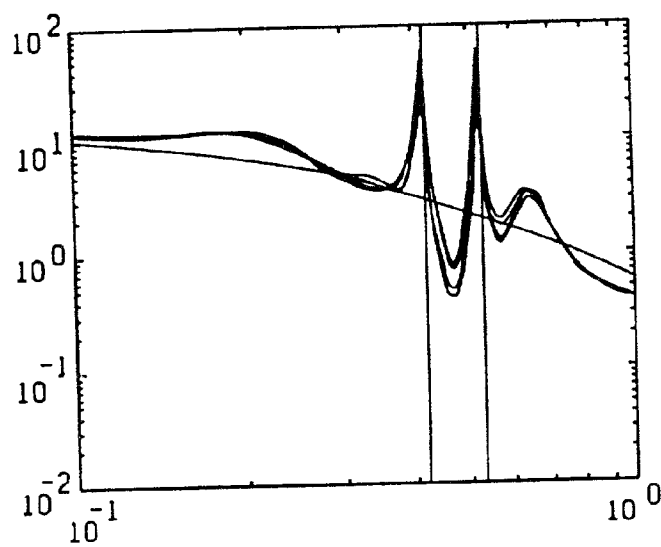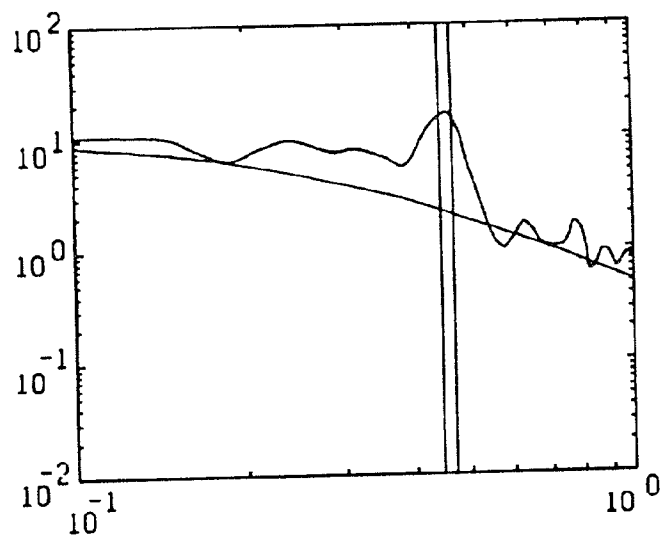
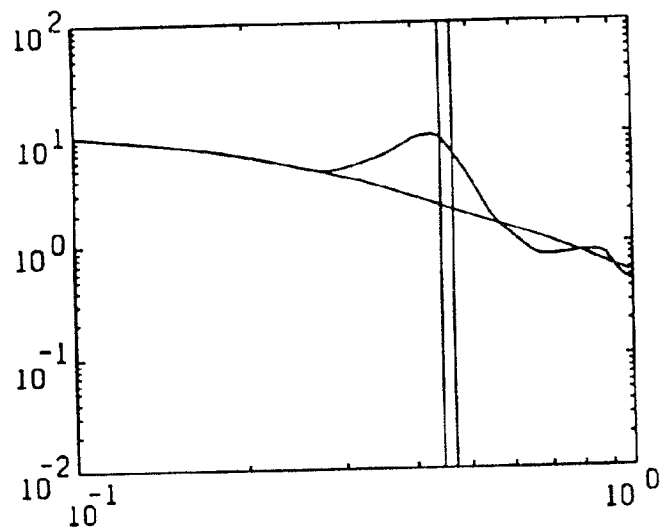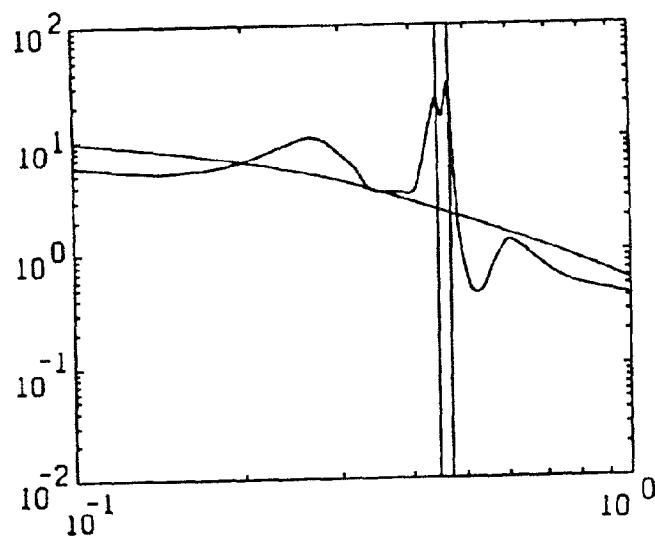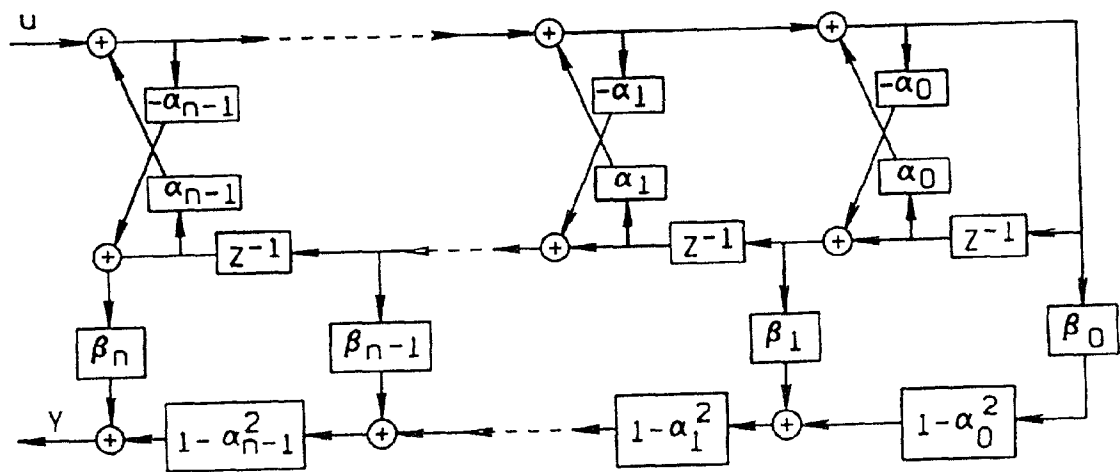FIG.4

FIG.5
PRIOR ART

FIG.6
PRIOR ART

FIG.7

FIG.8

FIG.9
PRIOR ART

FIG. 10

FIG. 11
PRIOR ART

FIG. 12

FIG. 13

Interface with Encoder

4". Filter Bank Parameters

7. Decoder

6". MA Parameters

10. Spectral Plotter

FIG. 14
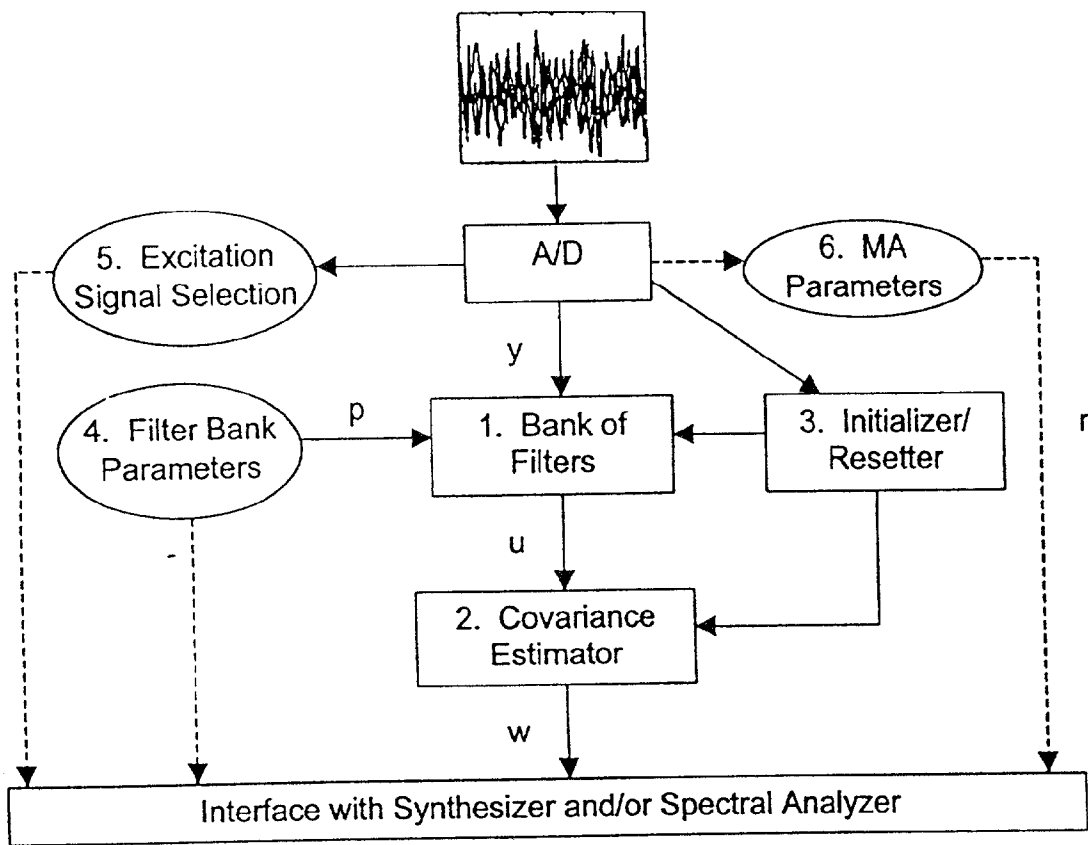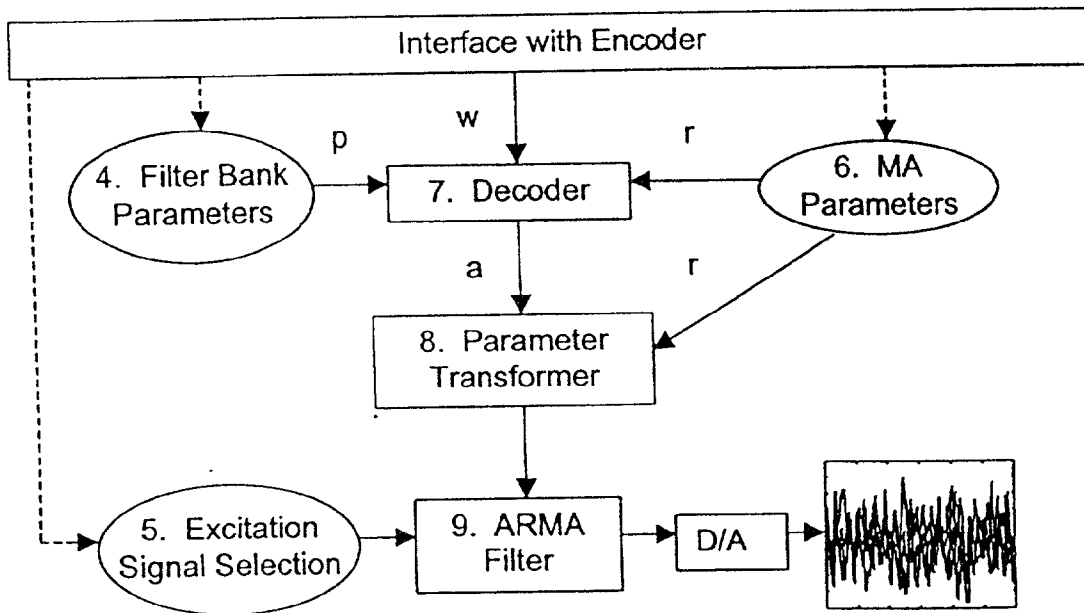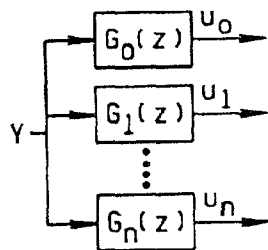
FIG. 15

FIG. 16

FIG. 17

FIG. 18

FIG. 19

FIG. 20

FIG. 21

METHOD AND APPARATUS FOR A TUNABLE HIGH-RESOLUTION SPECTRAL ESTIMATOR

BACKGROUND OF THE INVENTION

[0001]   We disclose a new method and apparatus for encoding and decoding signals and for performing high resolution spectral estimation. Many devices used in communications employ such devices for data compression, data transmission and for the analysis and processing of signals. The basic capabilities of the invention pertain to all areas of signal processing, especially for spectral analysis based on short data records or when increased resolution over desired frequency bands is required. One such filter frequently used in the art is the Linear Predictive Code (LPC) filter. Indeed, the use of LPC filters in devices for digital signal processing (see, e.g., U.S. Pat. Nos. 4,209,836 and 5,048,088 and D. Quarmby, *Signal Processing Chips*, Prentice Hall, 1994, and L. R. Rabiner, B. S. Atal, and J. L. Flanagan, *Current methods of digital speech processing*, Selected Topics in Signal Processing (S. Haykin, editor), Prentice Hall, 1989, 112-132) is pertinent prior art to the alternative which we shall disclose.

[0002]   We now describe this available art, the difference between the disclosed invention and this prior art, and the principal advantages of the disclosed invention. **FIG. 1** depicts the power spectrum of a sample signal, plotted in logarithmic scale.

[0003]   We have used standard methods known to those of ordinary skill in the art to develop a 4th order LPC filter from a finite window of this signal. The power spectrum of this LPC filter is depicted in **FIG. 2**.

[0004]   One disadvantage of the prior art LPC filter is that its power spectral density cannot match the "valleys," or "notches," in a power spectrum, or in a periodogram. For this reason encoding and decoding devices for signal transmission and processing which utilize LPC filter design result in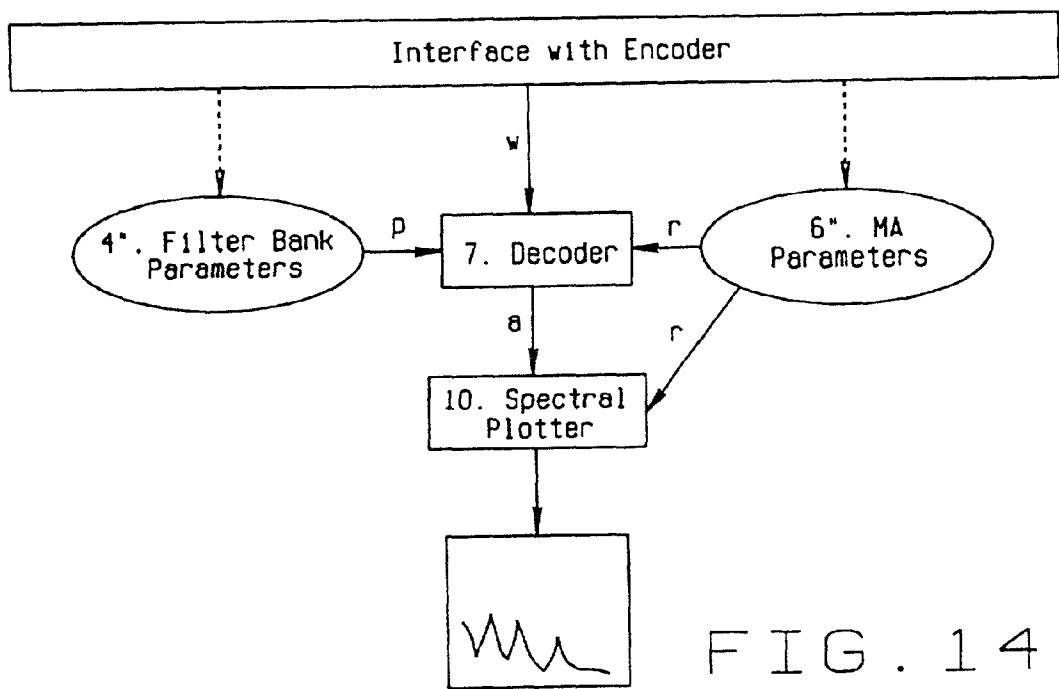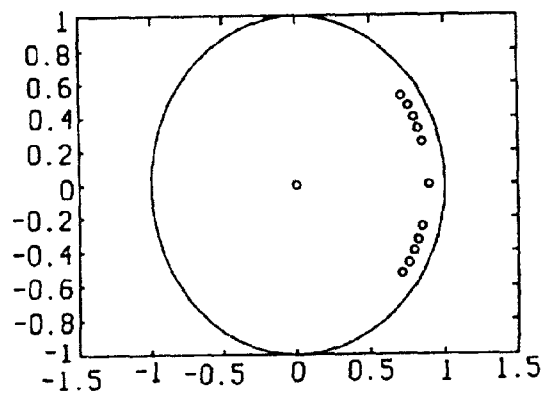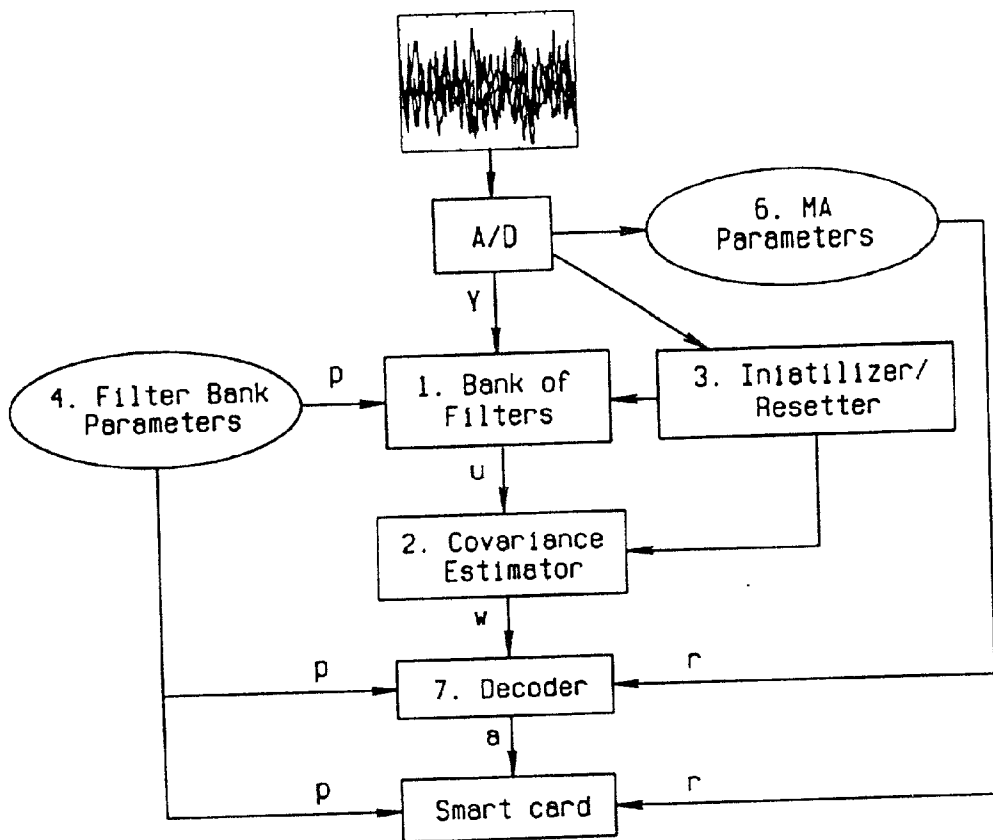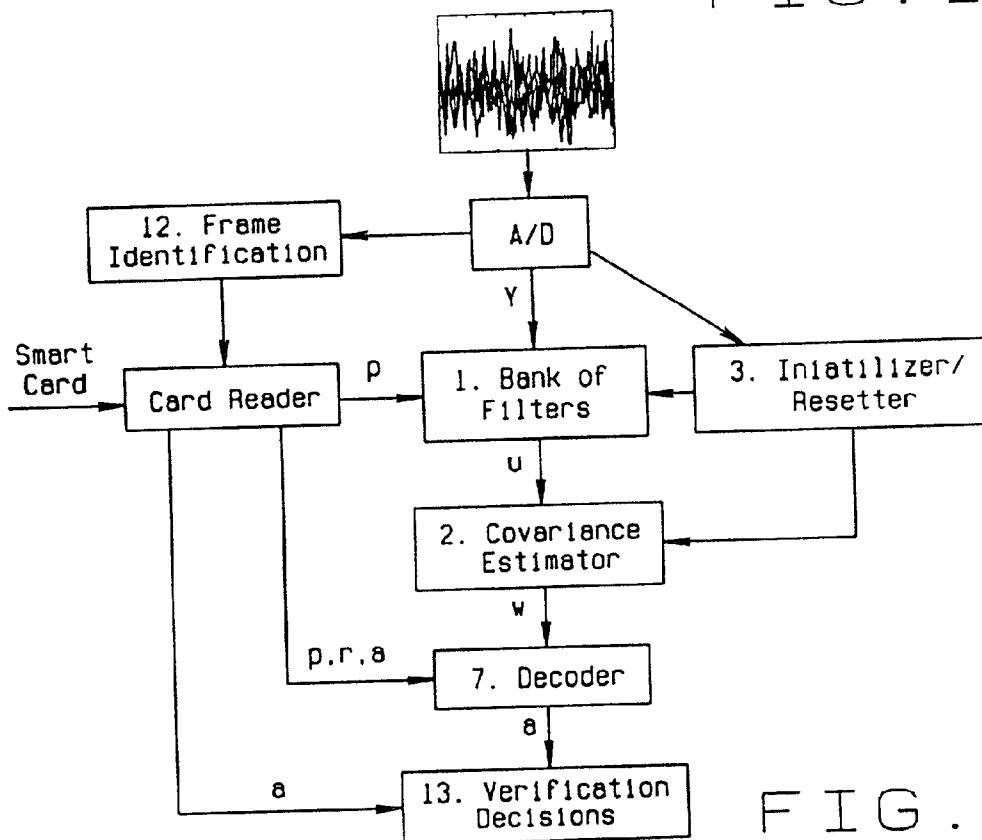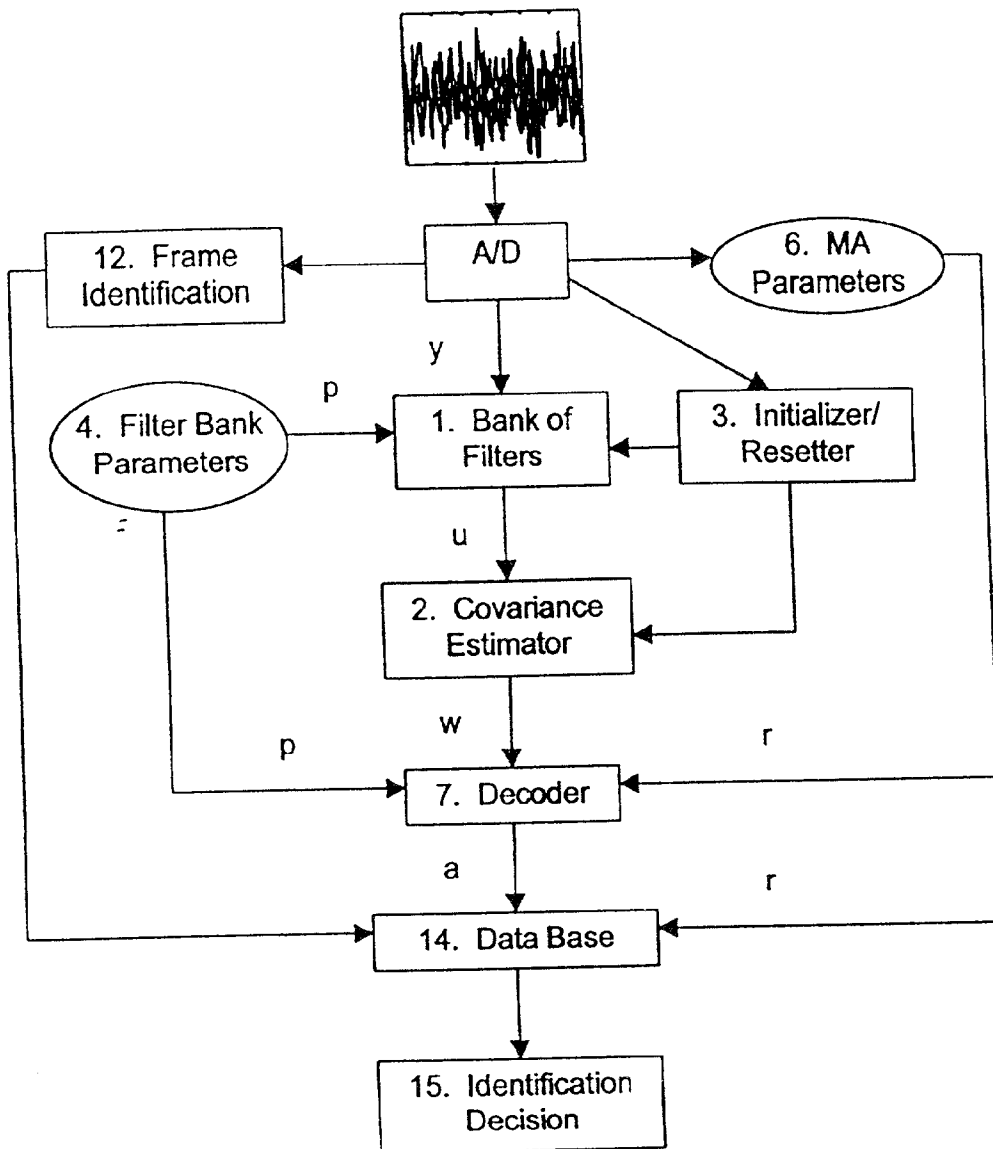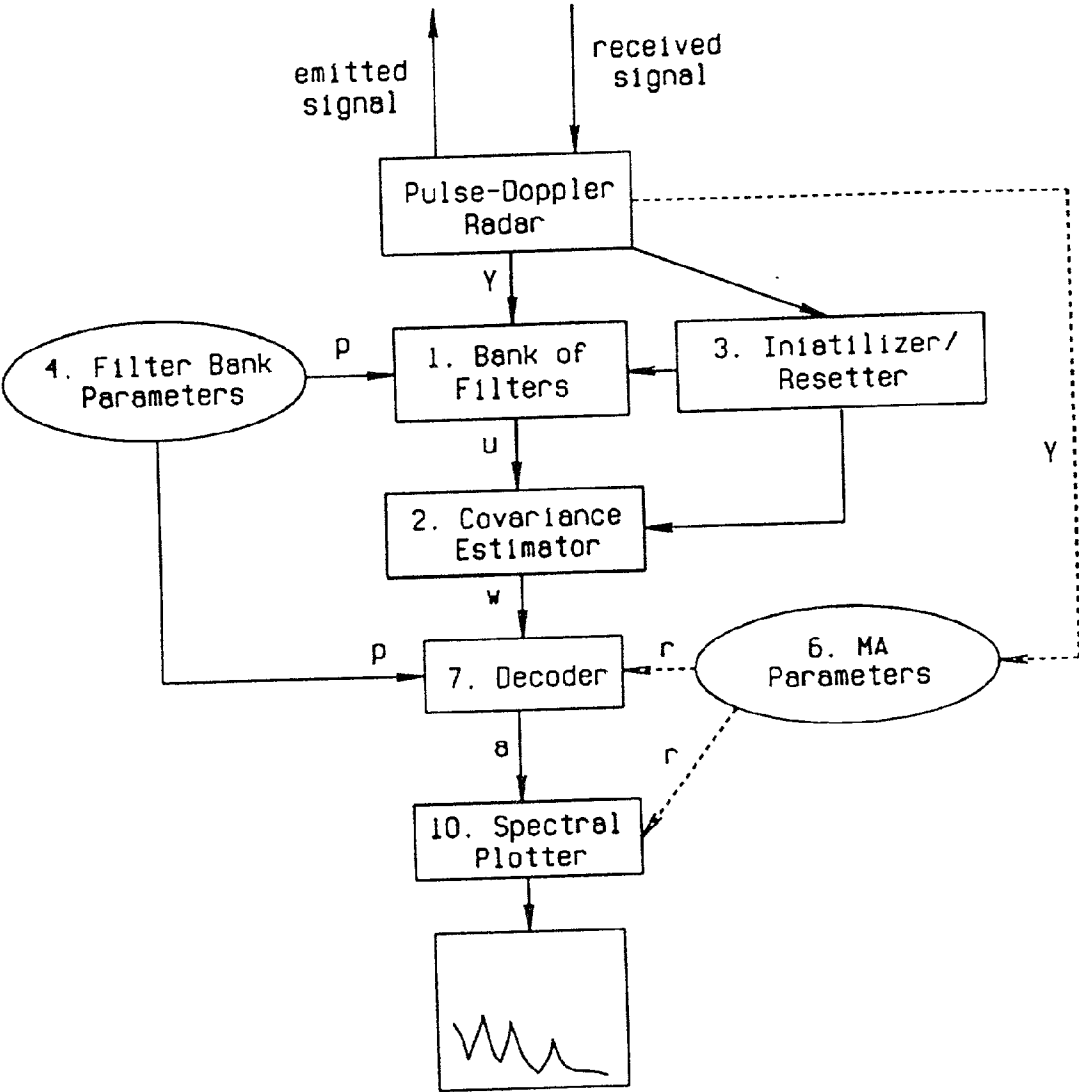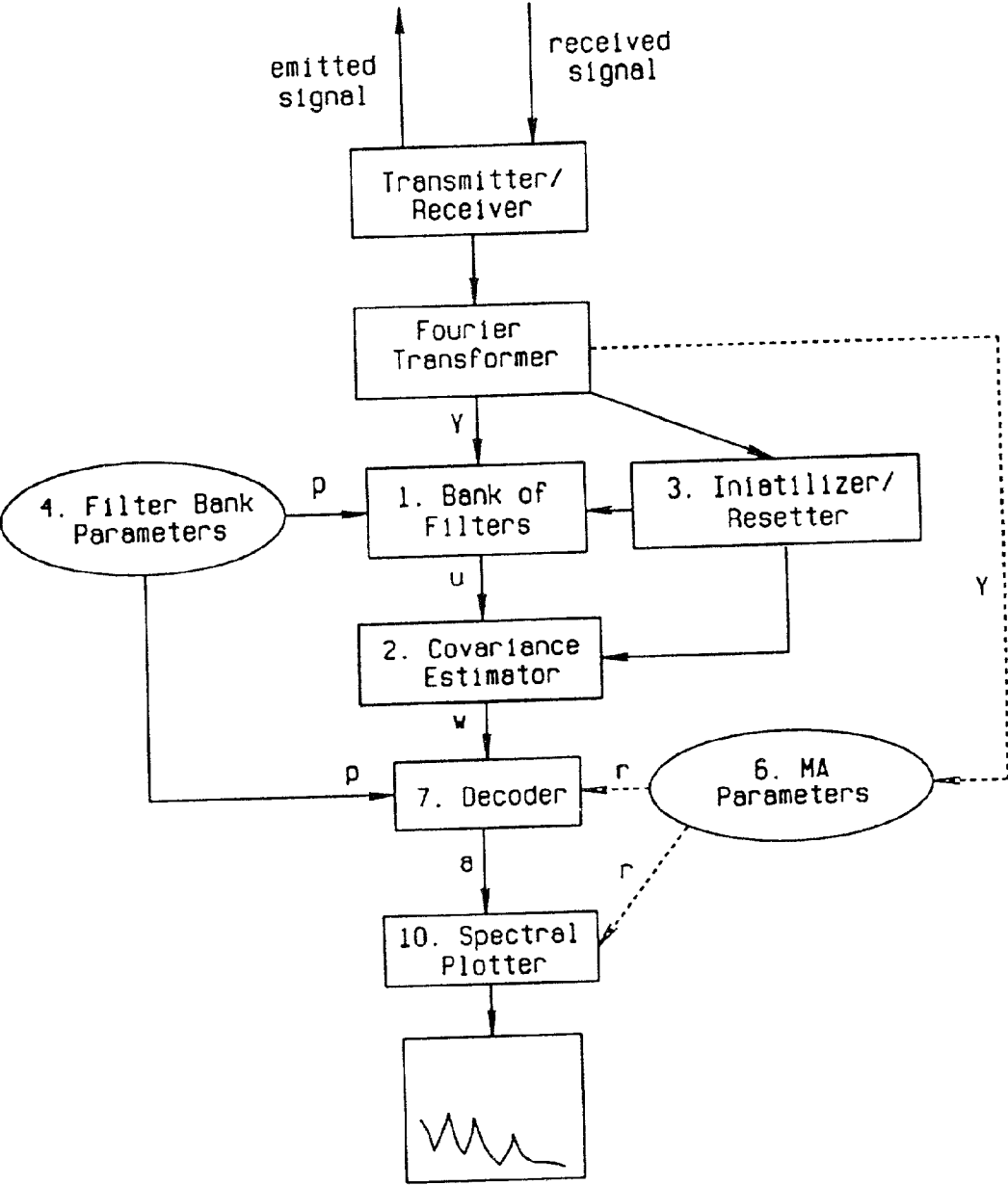 a synthesized signal which is rather "flat," reflecting the fact that the LPC filter is an "all-pole model." Indeed, in the signal and speech processing literature it is widely appreciated that regeneration of human speech requires the design of filters having zeros, without which the speech will sound flat or artificial; see, e.g., [C. G. Bell, H. Fujisaaki, J. M. Heinz, K. N. Stevons and A. S. House, *Reduction of Speech Spectra by Analysis-by-Synthesis Techniques,* J. Acoust. Soc. Am. 33 (1961), page 1726], [J. D. Markel and A. H. Gray, *Linear Prediction of Speech*, Springer Verlag, Berlin, 1976, pages 271-272], [L. R. Rabiner and R. W. Schafer, *Digital Processing of Speech Signals*, Prentice-Hall, Englewood Cliffs, N.J., 1978, pages 105, 76-78]. Indeed, while all pole filters can reproduce much of human speech sounds, the acoustic theory teaches that nasals and fricatives require both zeros and poles [J. D. Markel and A. H. Gray, *Linear Prediction of Speech*, Springer Verlag, Berlin, 1976, pages 271-272], [L. R. Rabiner and R. W. Schafer, *Digital Processing of Speech Signals*, Prentice-Hall, Englewood Cliffs, N.J., 1978, page 105]. This is related to the technical fact that the LPC filter only has poles and has no transmission zeros. To say that a filter has a transmission zero at a frequency ζ is to say that the filter, or corresponding circuit, will absorb damped periodic signals which oscillate at a frequency equal to the phase of ζ and with a damping factor equal to the modulus of ζ. This is the well-known blocking

property of transmission zeros of circuits, see for example [L. O. Chua, C. A. Desoer and E. S. Kuh, *Linear and Nonlinear Circuits*, McGraw-Hill, 1989, page 659]. This is reflected in the fact, illustrated in **FIG. 2**, that the power spectral density of the estimated LPC filter will not match the power spectrum at "notches," that is, frequencies where the observed signal is at its minimum power. Note that in the same figure the true power spectrum is indicated by a dotted line for comparison.

[0005]   Another feature of linear predictive coding is that the LPC filter reproduces a random signal with the same statistical parameters (covariance sequence) estimated from the finite window of observed data. For longer windows of data this is an advantage of the LPC filter, but for short data records relatively few of the terms of the covariance sequence can be computed robustly. This is a limiting factor of any filter which is designed to match a window of covariance data. The method and apparatus we disclose here incorporates two features which are improvements over these prior art limitations: The ability to include "notches" in the power spectrum of the filter, and the design of a filter based instead on the more robust sequence of first covariance coefficients obtained by passing the observed signal through a bank of first order filters. The desired notches and the sequence of (first-order) covariance data uniquely determine the filter parameters. We refer to such a filter as a tunable high resolution estimator, or THREE filter, since the desired notches and the natural frequencies of the bank of first order filters are tunable. A choice of the natural frequencies of the bank of filters correspond to the choice of a band of frequencies within which one is most interested in the power spectrum, and can also be automatically tuned. **FIG. 3** depicts the power spectrum estimated from a particular choice of 4th order THREE filter for the same data used to generate the LPC estimate depicted in **FIG. 2**, together with the true power spectrum, depicted in **FIG. 1**, which is marked with a dotted line.

[0006]   We expect that this invention will have application as an alternative for the use of LPC filter design in other areas of signal processing and statistical prediction. In particular, many devices used in communications, radar, sonar and geophysical seismology contain a signal processing apparatus which embodies a method for estimating how the total power of a signal, or (stationary) data sequence, is distributed over frequency, given a finite record of the sequence. One common type of apparatus embodies spectral analysis methods which estimate or describe the signal as a sum of harmonics in additive noise [P. Stoica and R. Moses, *Introduction to Spectral Analysis*, Prentice-Hall, 1997, page 139]. Traditional methods for estimating such spectral lines are designed for either white noise or no noise at all and can illustrate the comparative effectiveness of THREE filters with respect to both non-parametric and parametric based spectral estimation methods for the problem of line spectral estimation. **FIG. 4** depicts five runs of a signal comprised of the superposition of two sinusoids with colored noise, the number of sample points for each being 300. **FIG. 5** depicts the five corresponding periodograms computed with state-of-the-art windowing technology. The smooth curve represents the true power spectrum of the colored noise, and the two vertical lines the position of the sinusoids.

[0007]   **FIG. 6** depicts the five corresponding power spectra obtained through LPC filter design, while **FIG. 7** depicts

the corresponding power spectra obtained through the THREE filter design. **FIGS. 8, 9** and **10** show similar plots for power spectra estimated using state-of-the-art periodogram, LPC, and our invention, respectively. It is apparent that the invention disclosed herein is capable of resolving the two sinusoids, clearly delineating their position by the presence of two peaks. We also disclose that, even under ideal noise conditions the periodogram cannot resolve these two frequencies. In fact, the theory of spectral analysis [P. Stoica and R. Moses, *Introduction to Spectral Analysis*, Prentice-Hall, 1997, page 33] teaches that the separation of the sinusoids is smaller than the theoretically possible distance that can be resolved by the periodogram using a 300 point record under ideal noise conditions, conditions which are not satisfied here. This example represents a typical situation in applications.

[0008] The broader technology of the estimation of sinusoids in colored noise has been regarded as difficult [B. Porat, Digital *Processing of Random Signals*, Prentice-Hall, 1994, pages 285-286]. The estimation of sinusoids in colored noise using autoregressive moving-average filters, or ARMA models, is desirable in the art. As an ARMA filter, the THREE filter therefore possesses "super-resolution" capabilities [P. Stoica and R. Moses, *Introduction to Spectral Analysis*, Prentice-Hall, 1997, page 136].

[0009] We therefore disclose that the THREE filter design leads to a method and apparatus, which can be readily implemented in hardware or hardware/software with ordinary skill in the art of electronics, for spectral estimation of sinusoids in colored noise. This type of problem also includes time delay estimation [M. A. Hasan and M. R. Asimi-Sadjadi, *Separation of multiple time delays in using new spectral estimation schemes*, IEEE Transactions on Signal Processing 46 (1998), 2618-2630] and detection of harmonic sets [M. Zeytinoglu and K. M. Wong, *Detection of harmonic sets, IEEE Transactions on Signal Processing* 43 (1995), 2618-2630], such as in identification of submarines and aerospace vehicles. Indeed, those applications where tunable resolution of a THREE filter will be useful include radar and sonar signal analysis, and identification of spectral lines in doppler-based applications [P. Stoica and R. Moses, *Introduction to Spectral Analysis*, Prentice-Hall, 1997, page 248]. Other areas of potential importance include identification of formants in speech, data decimation [M. A. Hasan and M. R. Azimi-Sadjadi, *Separation of multiple time delays using new spectral estimation schemes*, IEEE Transactions on Signal Processing 46 (1998), 2618-2630], and nuclear magnetic resonance.

[0010] We also disclose that the basic invention could be used as a part of any system for speech compression and speech processing. In particular, in certain applications of speech analysis, such as speaker verification and speech recognition, high quality spectral analysis is needed [Joseph P. Campbell, Jr., *Speaker Recognition: A tutorial*, Proceedings of the IEEE 85 (1997), 1436-1463], [Jayant M. Naik, *Speaker Verification: A tutorial*, IEEE Communications Magazine, January 1990, 42-48], [Sadaoki Furui, *Recent advances in Speaker Recognition*, Lecture Notes in Computer Science 1206, 1997, 237-252], [Hiroaki Sakoe and Seibi Chiba, *Dynamic Programming Altorithm Optimization for Spoken Word Recognition*, IEEE Transactions on Acoustics, Speech and Signal Processing ASSP-26 (1978), 43-49]. The tuning capabilities of the device should prove especially

suitable for such applications. The same holds for analysis of biomedical signals such as EMG and EKG signals.

BRIEF DESCRIPTION OF THE DRAWINGS

[0011] **FIG. 1** is a graphical representation of the power spectrum of a sample signal;

[0012] **FIG. 2** is a graphical representation of the spectral estimate of the sample signal depicted in **FIG. 1** as best matched with an LPC filter;

[0013] **FIG. 3** is a graphical representation of the spectral estimate of the sample signal with true spectrum shown in **FIG. 1** (and marked with dotted line here for comparison), as produced with the invention;

[0014] **FIG. 4** is a graphical representation of five sample signals comprised of the superposition of two sinusoids with colored noise;

[0015] **FIG. 5** is a graphical representation of the five periodograms corresponding to the sample signals of **FIG. 4**;

[0016] **FIG. 6** is a graphical representation of the five corresponding power spectra obtained through LPC filter design for the five sample signals of **FIG. 4**;

[0017] **FIG. 7** is a graphical representation of the five corresponding power spectra obtained through the invention filter design;

[0018] **FIG. 8** is a graphical representation of a power spectrum estimated from a time signal with two closely spaced sinusoids (marked by vertical lines), using periodogram;

[0019] **FIG. 9** is a graphical representation of a power spectrum estimated from a time signal with two closely spaced sinusoids (marked by vertical lines), using LPC design;

[0020] **FIG. 10** is a graphical representation of a power spectrum estimated from a time signal with two closely spaced sinusoids (marked by vertical lines), using the invention;

[0021] **FIG. 11** is a schematic representation of a lattice-ladder filter in accordance with the present invention;

[0022] **FIG. 12** is a block diagram of a signal encoder portion of the present invention;

[0023] **FIG. 13** is a block diagram of a signal synthesizer portion of the present invention;

[0024] **FIG. 14** is a block diagram of a spectral analyzer portion of the present invention;

[0025] **FIG. 15** is a block diagram of a bank of filters, preferably first order filters, as utilized in the encoder portion of the present invention;

[0026] **FIG. 16** is a graphical representation of a unit circle indicating the relative location of poles for one embodiment of the present invention;

[0027] **FIG. 17** is a block diagram depicting a speaker verification enrollment embodiment of the present invention;

[0028] FIG. 18 is a block diagram depicting a speaker verification embodiment of the present invention;

[0029] FIG. 19 is a block diagram of a speaker identification embodiment of the present invention;

[0030] FIG. 20 is a block diagram of a doppler-based speed estimator embodiment of the present invention; and

[0031] FIG. 21 is a block diagram for a time delay estimator embodiment of the present invention.

[0032] The present invention of a THREE filter design retains two important advantages of linear predictive coding. The specified parameters (specs) which appear as coefficients (linear prediction coefficients) in the mathematical description (transfer function) of the LPC filter can be computed by optimizing a (convex) entropy functional. Moreover, the circuit, or integrated circuit device, which implements the LPC filter is designed and fabricated using ordinary skill in the art of electronics (see, e.g., U.S. Pat. Nos. 4,209,836 and 5,048,088) on the basis of the specified parameters (specs). For example, the expression of the specified parameters (specs) is often conveniently displayed in a lattice filter representation of the circuit, containing unit delays z^{-1}, summing junctions, and gains. The design of the associated circuit is well within the ordinary skill of a routineer in the art of electronics. In fact, this filter design has been fabricated by Texas Instruments, starting from the lattice filter representation (see, e.g., U.S. Pat. No. 4,344, 148), and is used in the LPC speech synthesizer chips TMS 5100, 5200, 5220 (see e.g. D. Quarmby, *Signal Processing Chips*, Prentice-Hall, 1994, pages 27-29).

[0033] In order to incorporate zeros as well as poles into digital filter models, it is customary in the prior art to use alternative architectures, for example the lattice-ladder architecture [K. J. Åström, *Evaluation of quadratic loss functions for linear systems*, in *Fundamentals of Discrete-time systems: A tribute to Professor Eliahu I. Jury*, M. Jamshidi, M. Mansour, and B.D.O. Anderson (editors), IITSI Press, Albuquerque, N. Mex., 1993, pp. 45-56] depicted in **FIG. 11**.

[0034] As for the lattice representation of the LPC filter, the lattice-ladder filter consists of gains, which are the parameter specs, unit delays z^{-1}, and summing junctions and therefore can be easily mapped onto a custom chip or onto any programmable digital signal processor (e.g., the Intel 2920, the TMS 320, or the NEC 7720) using ordinary skill in the art; see, e.g. D. Quarmby, *Signal Processing Chips*, Prentice-Hall, 1994, pages 27-29. We observe that the lattice-ladder filter representation is an enhancement of the lattice filter representation, the difference being the incorporation of the spec parameters denoted by β, which allow for the incorporation of zeros into the filter design. In fact, the lattice filter representation of an all-pole filter can be designed from the lattice-ladder filter architecture by setting the parameter specifications: $\beta_0 = r_n^{-\frac{1}{2}}$, $\beta_1 = \beta_2 = \ldots = \beta_n = 0$ and $\alpha_k = \gamma_k$ for k=0, 1, . . . , n−1 . We note that, in general, the parameters α_0, α_1, . . . , α_{n-1} are not the reflection coefficients (PARCOR parameters).

[0035] As part of this disclosure, we disclose a method and apparatus for determining the gains in a ladder-lattice embodiment of THREE filter from a choice of notches in the power spectrum and of natural frequencies for the bank of filters, as well as a method of automatically tuning these

notches and the natural frequencies of the filter bank from the observed data. Similar to the case of LPC filter design, the specs, or coefficients, of the THREE filter are also computed by optimizing a (convex) generalized entropy functional. One might consider an alternative design using adaptive linear filters to tune the parameters in the lattice-ladder filter embodiment of an autoregressive moving-average (ARMA) model of a measured input-output history, as has been done in [M. G. Bellanger, *Computational complexity and accuracy issues in fast least squares algorithms for adaptive filtering*, Proc. 1988 IEEE International Symposium on Circuits and Systems, Espoo, Finland, Jun. 7-9, 1988] for either lattice or ladder filter tuning. However, one should note that the input string which might generate the observed output string is not necessarily known, nor is it necessarily available, in all situations to which THREE filter methods apply (e.g., speech synthesis). For this reason, one might then consider developing a tuning method for the lattice-ladder filter parameters using a system identification scheme based on an autoregressive moving-average with exogenous variables (ARMAX). However, the theory of system identification teaches that these optimization schemes are nonlinear but nonconvex [T. Söderström and P. Stoica, *Systems Identification*, Prentice-Hall, New York, 1989, page 333, equations (9.47), and page 334, equations (9.48)]. Moreover, the theory teaches that there are examples where global convergence of the associated algorithms may fail depending on the choice of certain design parameters (e.g., forgetting factors) in the standard algorithm [T. Söderström and P. Stoica, op. cit., page 340, Example 9.6]—in sharp contrast to the convex minimization scheme we disclose for the lattice-ladder parameters realizing a THREE filter. In addition, ARMAX schemes will not necessarily match the notches of the power spectrum. Finally, we disclose here that our extensive experimentation with both methods for problems of formant identification show that ARMAX methods require significantly higher order filters to begin to identify formants, and also lead to the introduction of spurious formants, in cases where THREE filter methods converge quite quickly and reliably.

[0036] We now disclose a new method and apparatus for encoding and reproducing time signals, as well as for spectral analysis of signals. The method and apparatus, which we refer to as the Tunable High Resolution Estimator (THREE), is especially suitable for processing and analyzing short observation records.

[0037] The basic parts of the THREE are: the Encoder, the Signal Synthesizer, and the Spectral Analyzer. The Encoder samples and processes a time signal (e.g., speech, radar, recordings, etc.) and produces a set of parameters which are made available to the Signal Synthesizer and the Spectral Analyzer. The Signal Synthesizer reproduces the time signal from these parameters. From the same parameters, the Spectral Analyzer generates the power spectrum of the time-signal.

[0038] The design of each of these components is disclosed with both fixed-mode and tunable features. Therefore, an essential property of the apparatus is that the performance of the different components can be enhanced for specific applications by tuning two sets of tunable parameters, referred to as the filter-bank poles $p=(p_0, p_1, \ldots, p_n)$ and the MA parameters $r=(r_1, r_2, \ldots, r_n)$ respectively. In this disclosure we shall teach how the value

4

of these parameters can be (a) set to fixed "default" values, and (b) tuned to give improved resolution at selected portions of the power spectrum, based on a priori information about the nature of the application, the time signal, and statistical considerations. In both cases, we disclose what we believe to be the preferred embodiments for either setting or tuning the parameters.

[0039] As noted herein, the THREE filter is tunable. However, in its simplest embodiment, the tunable feature of the filter may be eliminated so that the invention incorporates in essence a high resolution estimator (HREE) filter. In this embodiment the default settings, or a priori information, is used to preselect the frequencies of interest. As can be appreciated by those of ordinary skill in the art, in many applications this a priori information is available and does not detract from the effective operation of the invention. Indeed the tunable feature is not needed for these applications. Another advantage of not utilizing the tunable aspect of the invention is that faster operation is achieved. This increased operational speed may be more important for some applications, such as those which operate in real time, rather than the increased accuracy of signal reproduction expected with tuning. This speed advantage is expected to become less important as the electronics available for implementation are further improved.

[0040] The intended use of the apparatus is to achieve one or both of the following objectives: (1) a time signal is analyzed by the Encoder and the set of parameters are encoded, and transmitted or stored. Then the Signal Synthesizer is used to reproduce the time signal; and/or (2) a time signal is analyzed by the Encoder and the set of parameters are encoded, and transmitted or stored. Then the Spectral Analyzer is used to identify the power spectrum of time signal over selected frequency bands.

[0041] These two objectives could be achieved in parallel, and in fact, data produced in conjunction with (2) may be used to obtain more accurate estimates of the MA parameters, and thereby improve the performance of the time synthesizer in objective (1). Therefore, a method for updating the MA parameters on-line is also disclosed.

[0042] The Encoder. Long samples of data, as in speech processing, are divided into windows or frames (in speech typically a few 10 ms.), on which the process can be regarded as being stationary. The procedure of doing this is well-known in the art [T. P. Barnwell III, K. Nayebi and C. H. Richardson, *Speech Coding: A Computer Laboratory Textbook*, John Wiley & Sons, New York, 1996]. The time signal in each frame is sampled, digitized, and de-trended (i.e., the mean value subtracted) to produce a (stationary) finite time series

$$y(0), y(1), \ldots, y(N). \tag{2.1}$$

[0043] This is done in the box designated as A/D in **FIG. 12**. This is standard in the art [T. P. Barnwell III, K. Nayebi and C. H. Richardson, *Speech Coding: A Computer Laboratory Textbook*, John Wiley & Sons, New York, 1996]. The separation of window frames is decided by the Initializer/Resetter, which is Component **3** in **FIG. 12**. The central component of the Encoder is the Filter Bank, given as component **1**. This consists of a collection of n+1 low-order filters, preferably first order filters, which process the observed time series in parallel. The output of the Filter

Bank consists of the individual outputs compiled into a time sequence of vectors

$$\begin{bmatrix} u_0(t_0) \\ u_1(t_0) \\ \vdots \\ u_n(t_0) \end{bmatrix}, \begin{bmatrix} u_0(t_0+1) \\ u_1(t_0+1) \\ \vdots \\ u_n(t_0+1) \end{bmatrix}, \ldots, \begin{bmatrix} u_0(N) \\ u_1(N) \\ \vdots \\ u_n(N) \end{bmatrix} \tag{2.2}$$

[0044] The choice of starting point t_0 will be discussed in the description of Component **2**.

[0045] As will be explained in the description of Component **7**, the Filter Bank is completely specified by a set $p=(p_0, p_1, \ldots, p_n)$ of complex numbers. As mentioned above, these numbers can either be set to default values, determined automatically from the rules disclosed below, or tuned to desired values, using an alternative set of rules which are also disclosed below. Component **2** in **FIG. 12**, indicated as Covariance Estimator, produces from the sequence u(t) in (2.2) a set of n+1 complex numbers

$$w=(w_0, w_1, \ldots, w_n) \tag{2.3}$$

[0046] which are coded and passed on via a suitable interface to the Signal Synthesizer and the Spectral Analyzer. It should be noted that both sets p and w are self-conjugate. Hence, for each of them, the information of their actual values is carried by n+1 real numbers.

[0047] Two additional features which are optional, are indicated in **FIG. 12** by dashed lines. First, Component **5**, designated as Excitation Signal Selection, refers to a class of procedures to be discussed below, which provide the modeling filter (Component **9**) of the signal Synthesizer with an appropriate input signal. Second, Component **6**, designated as MA Parameters in **FIG. 12**, refers to a class of procedures for determining n real numbers

$$r=(r_1, r_2, \ldots, r_n), \tag{2.4}$$

[0048] the so-called MA parameters, to be defined below.

[0049] The Signal Synthesizer. The core component of the Signal Synthesizer is the Decoder, given as Component **7** in **FIG. 13**, and described in detail below. This component can be implemented in a variety of ways, and its purpose is to integrate the values w, p and r into a set of n+1 real parameters

$$a=(a_0, a_1, \ldots, a_n), \tag{2.5}$$

[0050] called the AR parameters. This set along with parameters r are fed into Component **8**, called Parameter Transformer in **FIG. 13**, to determine suitable ARMA parameters for Component **9**, which is a standard modeling filter to be described below. The modeling filter is driven by an excitation signal produced by Component **5'**.

[0051] The Spectral Analyzer. The core component of the Spectral Analyzer is again the Decoder, given as Component **7** in **FIG. 14**. The output of the Decoder is the set of AR parameters used by the ARMA modeling filter (Component **10**) for generating the power spectrum. Two optional features are driven by the Component **10**. Spectral estimates can be used to identify suitable updates for the MA parameters and/or updates of the Filter Bank parameters. The latter

5

option may be exercised when, for instance, increased resolution is desired over an identified frequency band.

[0052] Components. Now described in detail are the key components of the parts and their function. They are discussed in the same order as they have been enumerated in FIGS. 12-14.

[0053] Bank of Filters. The core component of the Encoder is a bank of n+1 filters with transfer functions

$$G_k(z) = \frac{z}{z - p_k}, \quad k = 0, 1, 2, \ldots ,$$

[0054] where the filter-bank poles p_0, p_1, . . . , p_n) are available for tuning. The poles are taken to be distinct and one of them, p_0 at the origin, i.e. $p_0=0$. As shown in FIG. 15, these filters process in parallel the input time series (2.1), each yielding an output U_k satisfying the recursion

$$u_k(t) = p_k u_k(t-1) + y(t). \tag{2.6}$$

[0055] Clearly, $u_0=y$. If p_k is a real number, this is a standard first-order filter. If p_k is complex,

$$u_k(t) := \xi_k(t) + i\eta_k(t)$$

[0056] can be obtained via the second order filter

$$\begin{bmatrix} \xi_k(t) \\ \eta_k(t) \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} \xi_k(t-1) \\ \eta_k(t-1) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} y(t), \tag{2.7}$$

[0057] where $p_k=a+ib$. Since complex filter-bank poles occur in conjugate pairs a±ib, and since the filter with the pole $p_l=a-ib$ produces the output

$$u_k(t) := \xi_k(t) - i\eta_k(t)$$

[0058] the same second order filter (2.7) replaces two complex one-order filters. We also disclose that for tunability of the apparatus to specific applications there may also be switches at the input buffer so that one or more filters in the bank can be turned off. The hardware implementation of such a filter bank is standard in the art.

[0059] The key theoretical idea on which our design relies, described in C. I. Byrnes, T. T. Georgiou, and A. Lindquist, *A new approach to Spectral Estimation: A tunable high-resolution spectral estimator*, preprint, is the following: Given the unique proper rational function f(z) with all poles in the unit disc $\{z\|z\|<1\}$ such that

$$\Phi(e^{i\theta}) := f(e^{i\theta}) + f(e^{-i\theta}), \quad -\pi \leq \theta \leq \pi \tag{2.8}$$

[0060] is the power spectrum of y, it can be shown that

$$f(p_k^{-1}) = \frac{1}{2}(1 - p_k^2)E\{u_k(t)^2\}, \quad t \geq t_0, \tag{2.9}$$

[0061] where $E\{\cdot\}$ is mathematical expectation, provided t_0 is chosen large enough for the filters to have reached steady state so that (2.2) is a stationary process; see C. I. Byrnes, T. T. Georgiou, and A. Lindquist, *A new approach*

to Spectral Estimation: A tunable high-resolution spectral estimator, preprint. The idea is to estimate the variances

$$c_0(u_k) := E\{u_k(t)^2\}, \quad k=0, 1, \ldots, n$$

[0062] from output data, as explained under point 2 below, to yield interpolation conditions

$$f(Z_k) = w_k, \quad k=0, 1, \ldots, n \text{ where } z_k = p_k^{-1}$$

[0063] from which the function f(z), and hence the power spectrum Φ can be determined. The theory described in C. I. Byrnes, T. T. Georgiou, and A. Lindquist, *A new approach to Spectral Estimation: A tunable high-resolution spectral estimator*, preprint teaches that there is not a unique such f(z), and our procedure allows for making a choice which fulfills other design specifications.

[0064] Covariance Estimator. Estimation of the variance

$$c_0(v) := E\{v(t)^2\}$$

[0065] of a stationary stochastic process v(t) from an observation record

$$v_0, v_1, v_2, \ldots, v_N$$

[0066] can be done in a variety of ways. The preferred procedure is to evaluate

$$\hat{c}_0(v) := \frac{1}{N+1} \sum_{t=0}^{N} v_t^2 \tag{2.10}$$

[0067] over the available frame.

[0068] In the present application, the variances $\hat{c}_0(u_0)$, $\hat{c}_0(u_1)$. . . , $\hat{c}_0(u_n)$ are estimated and the numbers (2.3) are formed as

$$w_k := \frac{1}{2}(1 - p_k^2)\hat{c}_0(u_k), \quad k = 0, 1, \ldots, n. \tag{2.11}$$

[0069] Complex arithmetic is preferred, but, if real filter parameters are desired, the output of the second-order filter (2.7) can be processed by noting that

$$c_0(u_k) := c_0(\xi_k) - c_0(\eta_k) + 2i \, \mathrm{cov}(\xi_k, \eta_k),$$

[0070] where $\mathrm{cov}(\xi_k, \eta_k) := E\{\xi(t)\eta_k(t)\}$ is estimated by a mixed ergodic sum formed in analogy with (2.10).

[0071] Before delivering w=(w_0, w_1, . . . , w_n) as the output, check that the Pick matrix

$$P = \left[\frac{w_j + \overline{w}_k}{1 - p_j \overline{p}_k} \right]_{j,k=0}^{n}$$

[0072] is positive definite. If not, exchange w_k for $w_k+\lambda$ for k=0, 1, . . . , n, where λ is larger than the absolute value of the smallest eigenvalue of PP_0^{-1} where

$$P_0 = \left[\frac{2}{1 - p_j \overline{p}_k} \right]_{j,k=0}^{n}$$

[0073] Initializer/Resetter. The purpose of this component is to identify and truncate portions of an incoming time

series to produce windows of data (2.1), over which windows the series is stationary. This is standard in the art [T. P. Barnwell III, K. Nayebi and C. H. Richardson, *Speech Coding: A Computer Laboratory Textbook*, John Wiley & Sons, New York, 1996]. At the beginning of each window it also initializes the states of the Filter Bank to zero, as well as resets summation buffers in the Covariance Estimator (Component **2**).

[0074] Filter Bank Parameters. The theory described in C. I. Byrnes, T. T. Georgiou, and A. Lindquist, *A new approach to Spectral Estimation: A tunable high-resolution spectral estimator*, preprint, requires that the pole of one of the filters in the bank be at z=0 for normalization purposes; we take this to be p_0. The location of the poles of the other filters in the bank represents a design trade-off. The presence of Filter Bank poles close to a selected arc $\{e^{i\theta}|\theta \in [\theta_1, \theta_2]\}$ of the unit circle, allows for high resolution over the corresponding frequency band. However, proximity of the poles to the unit circle may be responsible for deterioration of the variability of the covariance estimates obtained by Component **2**.

[0075] There are two observations which are useful in addressing the design trade-off. First, the size n of the data bank is dictated by the quality of the desired reproduction of the spectrum and the expected complexity of it. For instance, if the spectrum is expected to have k spectral lines or formants within the targeted frequency band, typically, a filter of order n=2k+2 is required for reasonable reproduction of the characteristics.

[0076] Second, if N is the length of the window frame, a useful rule of thumb is to place the poles within

$$|p| < 10^{-\frac{10}{N}}.$$

[0077] This guarantees that the output of the filter bank attains stationarity in about 1/10 of the length of the window frame. Accordingly the Covariance Estimator may be activated to operate on the later 90% stationary portion of the processed window frame. Hence, t_0 in (2.2) can be taken to be the smallest integer larger than

$$\frac{N}{10}.$$

[0078] This typically gives a slight improvement as compared to the Covariance Estimator processing the complete processed window frame.

[0079] There is a variety of ways to take advantage of the design trade-offs. We now disclose what we believe are the best available rules to automatically determine a default setting of the bank of filter poles, as well as to automatically determine the setting of the bank of filter poles given a priori information on a bandwidth of frequencies on which higher resolution is desired.

[0080] Default Values.

[0081] (a) One pole is chosen at the origin,

[0082] (b) choose one or two real poles at

$$p = \pm 10^{\frac{10}{N}}$$

[0083] (c) choose an even number of equally spaced poles on the circumference of a circle with radius

$$10^{-\frac{10}{N}},$$

[0084] in a Butterworth-like pattern with angles spanning the range of frequencies where increased resolution is desired.

[0085] The total number of elements in the filter bank should be at least equal to the number suggested earlier, e.g., two times the number of formants expected in the signal plus two.

[0086] In the tunable case, it may be necessary to switch off one or more of the filters in the bank.

[0087] As an illustration, take the signal of two sinusoidal components in colored noise depicted in **FIG. 4**. More specifically, in this example,

$$y(t)=0.5 \sin (\omega_1 t+\phi_1)+0.5 \sin (\omega_2 t+\phi_2)+z(t) t=0,1,2,\ldots,$$
$$z(t)=0.8z(t-1)+0.5v(t)+0.25v(t-1)$$

[0088] with $\omega_1=0.42, \omega_2=0.53$, and ϕ_1, ϕ_2 and $v(t)$ independent N(0,1) random variables, i.e., with zero mean and unit variance. The squares in **FIG. 16** indicate suggested position of filter bank poles in order to attain sufficient resolution over the frequency band [0.4 0.5] so as to resolve spectral lines situated there and indicated by 0. The position of the poles on the circle |z|=0.9 is dictated by the length N~**300** for the time series window.

[0089] A THREE filter is determined by the choice of filter-bank poles and a choice of MA parameters. The comparison of the original line spectra with the power spectrum of the THREE filter determined by these filter-bank poles and the default value of the MA parameters, discussed below, is depicted in **FIG. 7**.

[0090] Excitation Signal Selection. An excitation signal is needed in conjunction with the time synthesizer and is marked as Component 5'. For some applications the generic choice of white noise may be satisfactory, but in general, and especially in speech it is a standard practice in vocoder design to include a special excitation signal selection. This is standard in the art [T. P. Barnwell III, K. Nayebi and C. H. Richardson, *Speech Coding: A Computer Laboratory Textbook*, John Wiley & Sons, New York, 1996, page 101 and pages 129-132] when applied to LPC filters and can also be implemented for general digital filters. The general idea adapted to our situation requires the following implementation.

[0091] Component **5** in **FIG. 12** includes a copy of the time synthesizer. That is, it receives as input the values w, p, and r, along with the time series y. It generates the coefficients a of the ARMA model precisely as the decoding section of the time synthesizer. Then it processes the time

series through a filter which is the inverse of this ARMA modeling filter. The "approximately whitened" signal is compared to a collection of stored excitation signals. A code identifying the optimal matching is transmitted to the time synthesizer. This code is then used to retrieve the same excitation signal to be used as an input to the modeling filter (Component **9** in **FIG. 13**).

[0092] Excitation signal selection is not needed if only the frequency synthesizer is used.

[0093] MA Parameter Selection. As for the filter-bank poles, the MA parameters can either be directly tuned using special knowledge of spectral zeros present in the particular application or set to a default value. However, based on available data (2.1), the MA parameter selection can also be done on-line, as described in Appendix A.

[0094] There are several possible approaches to determining a default value. For example, the choice $r_1=r_2=\ldots=r_n=0$ produces a purely autoregressive (AR) model which, however, is different from the LPC filter since it interpolates the filter-bank data rather than matching the covariance lags of the original process.

[0095] We now disclose what we believe is the best available method for determining the default values of the MA parameters. Choose $r_1, r_2, \ldots, r_n$ so that

$$z^n+r_1z^{n-1}+\ldots+r=(z-p_1)(z-p_2)\ldots(z-p_n), \qquad (2.12)$$

[0096] which corresponds to the central solution, described in Section **3**. This setting is especially easily implemented, as disclosed below.

[0097] Decoder. Given p, w, and r, the Decoder determines n+1 real numbers

$$a_0, a_1, a_2, \ldots, a_n, \qquad (2.13)$$

[0098] with the property that the polynomial

$$\alpha(z):=a_0z^n+a_1z^{n-1}+\ldots+a_n$$

[0099] has all its roots less than one in absolute value. This is done by solving a convex optimization problem via an algorithm presented in papers C. I. Byrnes, T. T. Georgiou, and A. Lindquist, *A generalized entropy criterion for Nevanlinna-Pick interpolation: A convex optimization approach to certain problems in systems and control*, preprint, and C. I. Byrnes, T. T. Georgiou, and A. Lindquist, *A new approach to Spectral Estimation: A tunable high-resolution spectral estimator*, preprint. While our disclosure teaches how to determine the THREE filter parameters on-line in the section on the Decoder algorithms, an alternative method and apparatus can be developed off-line by first producing a look-up table. The on-line algorithm has been programmed in MATLAB, and the code is enclosed in the Appendix B.

[0100] For the default choice (2.12) of MA-parameters, a much simpler algorithm is available, and it is also presented in the section on the Decoder algorithms. The MATLAB code for this algorithm is also enclosed in the Appendix B.

[0101] Parameter Transformer. The purpose of Component **8** in **FIG. 13** is to compute the filter gains for a modeling filter with transfer function

$$R(z) = \frac{z^n + r_1 z^{n-1} + \ldots + r_n}{a_0 z^n + a_1 z^{n-1} + \ldots + a_n}, \qquad (2.14)$$

[0102] where $r_1, r_2, \ldots, r_n$ are the MA parameters delivered by Component **6** (as for the Signal Synthesizer) or Component 6' (in the Spectral Analyzer) and $a_0, a_1, \ldots, a_n$ delivered from the Decoder (Component **7**) This can be done in many different ways [L. A. Chua, C. A. Desoer and E. S. Kuh, *Linear and Nonlinear Circuits*, McGraw-Hill, 1989], depending on desired filter architecture.

[0103] A filter design which is especially suitable for an apparatus with variable dimension is the lattice-ladder architecture depicted in **FIG. 11**. In this case, the gain parameters

$$\alpha_0, \alpha_1, \ldots, \alpha_{n-1} \text{ and } \beta_0, \beta_1, \ldots, \beta_n$$

[0104] are chosen in the following way. For $k=n, n-1, \ldots, 1$, solve the recursions

$$
\begin{cases}
a_{k-1,j} = a_{kj} + \alpha_{k-1}a_{k,k-j}, \ a_{nj} = a_j \\
\alpha_{k-1} = -\dfrac{a_{kk}}{a_{k0}} \\
r_{k-1,j} = r_{kj} - \beta_k a_{k,k-j}, \ r_{nj} = r_j \\
\beta_k = \dfrac{r_{kk}}{a_{k0}}
\end{cases}
\qquad (2.15)
$$

[0105] for $j=0, 1, \ldots, k$, and set

$$\beta_0 = \frac{r_{00}}{a_{00}}.$$

[0106] This is a well-known procedure; see, e.g., K. J. Astrtöm, Introduction to stochastic realization theory, Academic Press, 1970; and K. J. Aström, *Evaluation of quadratic loss functions of linear systems, in Fundamentals of Discrete-time systems: A tribute to Professor Eliahu I. Jury*, M. Jarnshidi, M. Mansour, and B. D. O. Anderson (editors), IITSI Press, Albuquerque, N. Mex., 1993, pp. 45-56. The algorithm is recursive, using only ordinary arithmetic operations, and can be implemented with an MAC mathematics processing chip using ordinary skill in the art.

[0107] ARMA filter. An ARMA modeling filter consists of gains, unit delays z^{-1}, and summing junctions, and can therefore easily be mapped onto a custom chip or any programmable digital signal processor using ordinary skill in the art. The preferred filter design, which easily can be adjusted to different values of the dimension n, is depicted in **FIG. 11**. If the AR setting $r_1=r_2=\ldots=r_n=0$ of the MA parameters has been selected, $\beta_0=r_n^{-1/2}$, $\beta_1=\beta_2=\ldots=\beta_n=0$ and $\alpha_k=\gamma_k$ for $k=0, 1, \ldots, n-1$, where $\gamma_k, k=0,1, \ldots, n-1$, are the first n PARCOR parameters and the algorithm (2.15) reduces to the Levinson algorithm [B. Porat, *digital Processing of Random Signals*, Prentice-Hall, 1994; and P. Stoica and R. Moses, Introduction to Spectral Analysis, Prentice-Hall, 1997].

[0108] Spectral plotter. The Spectral Plotter amounts to numerical implementation of the evaluation $\Phi(e^{i\theta}):=$

$|R(e^{i\theta})|^2$, where $R(z)$ is defined by (2.14), and θ ranges over the desired portion of the spectrum. This evaluation can be efficiently computed using standard FFT transform [P. Stoica and R. Moses, Introduction to Spectral Anqalysis, Prentice-Hall, 1997]. For instance, the evaluation of a polynomial (3.4) over a frequency range $z=e^{i\theta}$, with $\theta\in\{0, \Delta\theta, \ldots, 2\pi-\Delta\theta\}$ and $\Delta\theta=2\pi/M$, can be conveniently computed by obtaining the discrete Fourier transform of

$$(a_n, \ldots, a_1, 1, 0, \ldots, 0).$$

[0109] This is the coefficient vector padded with M-n-1 zeros. The discrete Fourier transform can be implemented using the FFT algorithm in standard form.

[0110] Decoder Algorithms. We now disclose the algorithms used for the Decoder. The input data consists of

[0111] (i) the filter-bank poles $p=(p_0, p_1, \ldots, p_n)$, which are represented as the roots of a polynomial

$$\tau(z): = \prod_{k=1}^{n} (z - p_k) = z^n + \tau_1 z^{n-1} + \ldots + \tau_{n-1} z + \tau_n, \tag{3.1}$$

[0112] (ii) the MA parameters $r=(r_1, r_2, \ldots, r_n)$, which are real numbers such that the polynomial

$$\rho(z)=z^n+r_1 z^{n-1}+ \ldots +r_{n-1}z+r_n \tag{3.2}$$

[0113] has all its roots less than one in absolute value, and

[0114] (iii) the complex numbers

$$w=(w_0, w_1, \ldots, w_n)$$

[0115] determined as (2.11) in the Covariance Estimator.

[0116] The problem is to find AR parameters $a=(a_0, a_1, \ldots, a_n)$, real numbers with the property that the polynomial

$$\alpha(z)=a_0 z^n+a_1 z^{n-1}+ \ldots +a_{n-1}z+a_n \tag{3.4}$$

[0117] has all its roots less than one in absolute value, such that

$$\left| \frac{\rho(e^{i\theta})}{\alpha(e^{i\theta})} \right|^2$$

[0118] is a good approximation of the power spectrum $\Phi(e^{i\theta})$ of the process y in some desired part of the spectrum $\theta\in[-\pi,\pi]$. More precisely, we need to determine the function $f(z)$ in (2.8). Mathematically, this problem amounts to finding a polynomial (3.4) and a corresponding polynomial

$$\beta(z)=b_0 z^n+b_1 z^{n-1}+ \ldots +b_{n-1}z+b_n, \tag{3.5}$$

[0119] satisfying

$$\alpha(z)\beta(z^{-1})+\beta(z)\alpha(z^{-1})=\rho(z)\rho(z^{-1}) \tag{3.6}$$

[0120] such that the rational function

$$f(z) = \frac{\beta(z)}{\alpha(z)} \tag{3.7}$$

[0121] satisfies the interpolation condition

$$f(z_k)=w_k, \quad k=0, 1, \ldots, n \text{ where } z_k=p_k^{-1}. \tag{3.8}$$

[0122] For this purpose the parameters p and r are available for tuning. If the choice of r corresponds to the default value, $r_k=\tau_k$ for $k=1, 2, \ldots, n$ (i.e., taking $\rho(z)=\tau(z)$), the determination of the THREE filter parameters is considerably simplified. The default option is disclosed in the next subsection. The method for determining the THREE filter parameters in the tunable case is disclosed in the subsection following the next. Detailed theoretical descriptions of the method, which is based on convex optimization, are given in the papers [C. I. Byrnes, T. T. Georgiou, and A. Lindquist, *A generalized entropy criterion for Nevanlinna-Pick interpolation: A convex optimization approach to certain problems in systems and control*, preprint, and C. I. Byrnes, T. T. Georgiou, and A. Lindquist, *A new approach to Spectral Estimation: A tunable high-resolution spectral estimator*, preprint].

[0123] The central solution algorithm for the default filter. In the special case in which the MA parameters $r=(r_1, r_2, \ldots, r_n)$ are set equal to the coefficients of the polynomial (3.1), i.e., when $\rho(z)=\tau(z)$, a simpler algorithm is available. Here we disclose such an algorithm which is particularly suited to our application. Given the filter-bank parameters $p_0, p_1, \ldots, p_n$ and the interpolation values $w_0, w_1, \ldots, w_n$ determine two sets of parameters $s_1, s_2, \ldots, s_n$ and $v_1, v_2, \ldots, v_n$ defined as

$$s_k = \frac{1 - p_k}{1 + p_k} \text{ and } v_k = \frac{1 - w_k / w_0}{1 + w_k / w_0} \quad k = 1, 2, \cdots, n$$

[0124] and the coefficients $\sigma_1, \sigma_2, \ldots, \sigma_n$ of the polynomial

$$\sigma(s)=(s-s_1)(s-s_2) \ldots (s-s_n)=s^n+\sigma_1 s^{n-1}+ \ldots +\sigma_n.$$

[0125] We need a rational function

$$p(s): = \frac{x_1 s^{n-1} + \cdots + x_n}{s^n + \sigma_1 s^{n-1} + \cdots + \sigma_n}$$

[0126] such that

$$p(s_k)=v_k \quad k=1, 2, \ldots, n,$$

[0127] and a realization $p(z)=c(sI-A)^{-1}b$, where

$$A = \begin{bmatrix} -\sigma_1 & -\sigma_2 & \cdots & -\sigma_{n-1} & -\sigma_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

$$c = [0 \ 0 \ \cdots \ 0 \ 1]$$

[0128] and the n-vector b remains to be determined. To this end, choose a (reindexed) subset $s_1, s_2, \ldots, s_m$ of the parameters $s_1, s_2, \ldots, s_n$, including one and only one s_k from each complex pair $(s_k, \bar{s}_k)$, and decompose the following complex Vandermonde matrix and complex vector into their real and imaginary parts:

9

$$\begin{bmatrix} s_1^{n-1} & s_1^{n-2} & \cdots & 1 \\ s_2^{n-1} & s_2^{n-2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ s_m^{n-1} & s_m^{n-2} & \cdots & 1 \end{bmatrix} = U_r + iU_i, \quad \begin{bmatrix} v_1\sigma(s_1) \\ v_2\sigma(s_2) \\ \vdots \\ v_m\sigma(s_m) \end{bmatrix} = u_r + iu_i.$$

[0129] Then, remove all zero rows from u_i and u_i to obtain U_t and U_t, respectively, and solve the n×n system

$$\begin{bmatrix} U_r \\ U_t \end{bmatrix} x = \begin{bmatrix} u_r \\ u_t \end{bmatrix}$$

[0130] for the n-vector x with components $x_1, x_2, \ldots, x_n$. Then, padding x with a zero entry to obtain the (n+1)-vector

$$\begin{bmatrix} 0 \\ x \end{bmatrix},$$

[0131] the required b is obtained by removing the last component of the (n+1)-vector

$$R^{-1}\begin{bmatrix} 0 \\ x \end{bmatrix},$$

[0132] where R is the triangular (n+1)×(n+1)-matrix

$$R = \begin{bmatrix} & & & & 1 \\ & & & 1 & \sigma_1 \\ & & 1 & \sigma_1 & \sigma_2 \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots \\ 1 & \sigma_1 & \sigma_2 & \cdots & \sigma_n \end{bmatrix},$$

[0133] where empty matrix entries denote zeros.

[0134] Next, with prime (') denoting transposition, solve the Lyapunov equations

$P_o A + A' P_0 = c'c$

$(A - P_o^{-1}c'c)P_c + P_c(A - P_o^{-1}c'c)' = bb'$

[0135] which is a standard routine, form the matrix

$N = (I - P_o P_c)^{-1},$

[0136] and compute the (n+1)-vectors $h^{(1)}, h^{(2)}, h^{(3)}$ and $h^{(4)}$ with components

[0137] $h_0^{(1)} = 1$, $h_k^{(1)} = cA^{k-1}P_o^{-1}Nc'$, $k = 1, 2, \ldots, n$

[0138] $h_0^{(2)} = 0$, $h_k^{(2)} = cA^{k-1}N'b$, $k = 1, 2, \ldots, n$

[0139] $h_0^{(3)} = 0$, $h_k^{(3)} = -b'P_o A^{k-1}P_o^{-1}Nc'$, $k = 1, 2, \ldots, n$

[0140] $h_1^{(4)} = 1$, $h_k^{(4)} = -b'P_o A^{k-1}N'b$, $k = 1, 2, \ldots, n$.

[0141] Finally, compute the (n+1)-vectors

[0142] $y^{(j)} = TRh^{(j)}$, j=1,2,3,4

[0143] with components $y_0^{(j)}, y_1^{(j)}, \ldots, y_n^{(j)}$, j=1,2,3,4, where T is the (n+1)×(n+1) matrix, the k: th column of which is the vector of coefficients of the polynomial

$(s+1)^{n-k}(s-1)^k$, for k=0, 1, \ldots, n,

[0144] starting with the coefficient of S^n and going down to the constant term, and R is the matrix defined above. Now form

$$\hat{\alpha}_k = \frac{1}{\sqrt{1-\mu^2}}[\mu(y_k^{(3)} + y_k^{(1)}) + (y_k^{(4)} + y_k^{(2)})], k = 0, 1, \cdots, n,$$

$$\hat{\beta}_k = \frac{w_0}{\sqrt{1-\mu^2}}[\mu(y_k^{(3)} - y_k^{(1)}) + (y_k^{(4)} - y_k^{(2)})], k = 0, 1, \cdots, n,$$

where

$$\mu = -\frac{y_0^{(2)}}{y_0^{(1)}}.$$

[0145] The (central) interpolant (3.7) is then given by

$$f(z) = \frac{\hat{\beta}(z)}{\hat{\alpha}(z)},$$

[0146] where $\hat{\alpha}(z)$ and $\hat{\beta}(z)$ are the polynomials

$\hat{\alpha}(z) = \hat{\alpha}_0 z^n + \hat{\alpha}_1 z^{n-1} + \ldots + \hat{\alpha}_n,$

$\hat{\beta}(z) = \hat{\beta}_0 z^n + \hat{\beta}_1 z^{n-1} + \ldots + \hat{\beta}_n.$

[0147] However, to obtain the $\alpha(z)$ which matches the MA parameters r=τ, $\hat{\alpha}(z)$ needs to be normalized by setting

$$\alpha(z) = \frac{1 + \tau_1^2 + \cdots + \tau_n^2}{2(\hat{\alpha}_0\hat{\beta}_0 + \hat{\alpha}_1\hat{\beta}_1 + \hat{\alpha}_n\hat{\beta}_n)}\hat{\alpha}(z).$$

[0148] This is the output of the central solver.

[0149] Convex optimization algorithm for the tunable filter. To initiate the algorithm, one needs to choose an initial value for a, or, equivalently, for $\alpha(z)$, to be recursively updated. We disclose two methods of initialization, which can be used if no other guidelines, specific to the application, are available.

[0150] Initialization method 1. Given the solution of the Lyapunov equation

$$S = A'SA + c'c, \tag{3.9}$$

where

$$A = \begin{bmatrix} -\tau_1 & -\tau_2 & \cdots & -\tau_{n-1} & -\tau_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \tag{3.10}$$

-continued

$$c = [0 \ 0 \ \cdots \ 0 \ 1], \tag{3.11}$$

form

$$\kappa = y' \begin{bmatrix} S & 0 \\ 0 & 1 \end{bmatrix} y, \ y = L_n^{-1} r,$$

[0151] where r is the column vector having the coefficients $1, r_1, \ldots, r_n$ of (3.2) as components and where

$$L_n = \begin{bmatrix} & & & & 1 \\ & & & 1 & \tau_1 \\ & & 1 & \tau_1 & \tau_2 \\ & \ddots & \ddots & \ddots & \vdots \\ 1 & \tau_1 & \tau_2 & \cdots & \tau_n \end{bmatrix}. \tag{3.12}$$

Then take

$$\alpha(z) = \sqrt{\frac{\kappa}{2w_0}} \, \tau(z)$$

[0152] as initial value.

[0153] Initialization method 2. Take

$$\alpha(z) = \left| \frac{1 + r_1 + \ldots + r_n}{1 + \tau_1 + \ldots + \tau_n} \right| \alpha_c(z),$$

[0154] where $\alpha_c(Z)$ is the α-polynomial obtained by first running the algorithm for the central solution described above.

[0155] Algorithm. Given the initial (3.4) and (3.1), solve the linear system of equations

$$\left(\begin{bmatrix} 1 & \cdots & \tau_{n-2} & \tau_{n-1} & \tau_n \\ \tau_1 & \cdots & \tau_{n-1} & \tau_n \\ \tau_2 & \cdots & \tau_n \\ \vdots & \ddots \\ \tau_n \end{bmatrix} + \begin{bmatrix} 1 & \tau_1 & \tau_2 & \cdots & \tau_n \\ & 1 & \tau_1 & \cdots & \tau_{n-1} \\ & & 1 & \cdots & \tau_{n-2} \\ & & & \ddots & \vdots \\ & & & & 1 \end{bmatrix} \right)$$

$$\begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} a_0^2 + a_1^2 + a_2^2 + \cdots + a_n^2 \\ a_0 a_1 + a_1 a_2 + a_{n-1} a_n \\ a_0 a_2 + a_1 a_3 + a_{n-2} a_n \\ \vdots \\ a_0 a_n \end{bmatrix}$$

[0156] for the column vector S with components s_0, $s_1, \ldots, s_n$. Then, with the matrix L_n given by (3.12), solve the linear system

$$L_n h = s$$

[0157] for the vector

$$h = \begin{bmatrix} h_n \\ h_{n-1} \\ \vdots \\ h_0 \end{bmatrix} \tag{3.13}$$

[0158] The components of h are the Markov parameters defined via the expansion

$$q(z) = \frac{\sigma(z)}{\tau(z)} = h_0 + h_1 z^{-1} + h_2 z^{-2} + h_3 z^{-3} + \cdots, \tag{3.14}$$

where

$$\sigma(z) := s_0 z^n + s_1 z^{n-1} + \cdots + s_n.$$

[0159] The vector (3.13) is the quantity on which iterations are made in order to update $\alpha(z)$. More precisely, a convex function J(q) presented in C. I. Byrnes, T. T. Georgiou, and A. Lindquist, *A generalized entropy criterion for Nevanlina-Pick interpolation: A convex optimization approach to certain problems in systems and control*, preprint, and C. I. Byrnes, T. T. Georgiou, and A. Lindquist, *A new approach to spectral estimation: A tunable high-resolution spectral estimator*, preprint, is minimized recursively over the region where

$$q(e^{i\theta}) + q(e^{-i\theta}) > 0, \text{ for } -\pi \leqq \theta \leqq \pi \tag{3.15}$$

[0160] This is done by upholding condition (3.6) while successively trying to satisfy the interpolation condition (3.8) by reducing the errors

$$e_k = w_k - f(p_k^{-1}), \ k = 0, 1, \ldots, n. \tag{3.16}$$

[0161] Each iteration of the algorithm consists of two steps. Before turning to these, some quantities, common to each iteration and thus computed off-line, need to be evaluated.

[0162] Given the MA parameter polynomial (3.2), let the real numbers $\pi_0, \pi_1, \ldots, \pi_n$ can be defined via the expansion

$$\rho(z)\rho(z^{-1}) = \pi_0 + \pi_1(z + z^{-1}) + \pi_2(z^2 + z^{-2}) + \ldots + \pi_n(z^n + z^{-n}).$$

[0163] Moreover, given a subset $p_1, p_2, \ldots, p_m$ of the filter-bank poles $p_1, p_2, \ldots, p_n$ obtained by only including one p_k in each complex conjugate pair $(p_k, \bar{p}_k)$, form the corresponding Vandermonde matrix

$$V := \begin{bmatrix} p_1^{-(n-1)} & p_1^{-(n-2)} & \cdots & p_1^{-1} & 1 \\ p_2^{-(n-1)} & p_2^{-(n-2)} & \cdots & p_2^{-1} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_m^{-(n-1)} & p_m^{-(n-2)} & \cdots & p_m^{-1} & 1 \end{bmatrix} = V_r + V_i, \tag{3.18}$$

[0164] together with its real part V_r and imaginary part V_i. Moreover, given an arbitrary real polynomial

$$\gamma(z) = g_0 z^m + g_1 z^{m-1} + \ldots + g_m, \tag{3.19}$$

[0165]   define the (n+1)×(m+1) matrix

$$M(\gamma) := \qquad (3.20)$$

$$\begin{bmatrix} g_0 & g_1 & \cdots & g_n & g_{n+1} & \cdots & g_m & & \\ & g_0 & g_1 & \cdots & g_n & g_{n+1} & \cdots & g_m & \\ & & \ddots & \ddots & & \ddots & \ddots & & \ddots \\ & & & g_0 & g_1 & \cdots & g_n & g_{n+1} & \cdots & g_m \end{bmatrix}.$$

[0166]   We compute off-line $M(\rho)$, $M(\tau^*\rho)$ and $M(\tau\rho)$, where ρ and τ are the polynomials (3.2) and (3.1) and $\tau^*(z)$ is the reversed polynomial

$$\tau^*(z) = \tau_n z^n + \tau_{n-1} z^{n-1} + \ldots + \tau_1 z + 1.$$

[0167]   Finally, we compute off-line L_n, defined by (3.12), as well as the submatrix L_{n-1}.

[0168]   Step 1. In this step the search direction of the optimization algorithm is determined. Given $\alpha(z)$, first find the unique polynomial (3.5) satisfying (3.6). Identifying coefficients of z^k, $k=0, 1, \ldots, n$, this is seen to be a (regular) system of n+1 linear equations in the n+1 unknown b_0, $b_1, \ldots, b_n$ namely

$$\left(\begin{bmatrix} a_0 & \cdots & a_{n-2} & a_{n-1} & a_n \\ a_1 & \cdots & a_{n-1} & a_n & \\ a_2 & \cdots & a_n & & \\ \vdots & \ddots & & & \\ a_n & & & & \end{bmatrix} + \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_n \\ & a_0 & a_1 & \cdots & a_{n-1} \\ & & a_0 & \cdots & a_{n-2} \\ & & & \ddots & \vdots \\ & & & & a_0 \end{bmatrix} \right) \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \vdots \\ \pi_n \end{bmatrix},$$

[0169]   where $\pi_0, \pi_1, \ldots, \pi_n$ are given by (3.17). The coefficient matrix is a sum of a Hankel and a Toeplitz matrix and there are fast and efficient ways of solving such systems [G. Heinig, P. Jankowski and K. Rost, *Fast Inversion Algorithms of Toeplitz-plus-Hankel Matrices*, Numerische Mathematik 52 (1988), 665-682]. Next, form the function

$$f(z) = \frac{\beta(z)}{\alpha(z)}.$$

[0170]   This is a candidate for an approximation of the positive real part of the power spectrum Φ as in (2.8).

[0171]   Next, we describe how to compute the gradient ∇J. Evaluate the interpolation errors (3.16), noting that $e_0 = w_0 - b_0/a_0$, and decompose the complex vector

$$\begin{bmatrix} (e_1 - e_0)\tau(p_1^{-1}) \\ (e_2 - e_0)\tau(p_2^{-1}) \\ \vdots \\ (e_n - e_0)\tau(p_n^{-1}) \end{bmatrix} = v_r + i v_i$$

[0172]   into its real part v_r and imaginary part v_i. Let V_r and V_i be defined by (3.18). Remove all zero rows from V_i and v_i to obtain V_t and v_t. Solve the system

$$\begin{bmatrix} V_r \\ V_t \end{bmatrix} x = \begin{bmatrix} v_r \\ v_t \end{bmatrix}$$

[0173]   for the column vector x and form the gradient as

$$\nabla J = 2 \begin{bmatrix} SL_{n-1}^{-1} x \\ 2e_0 \end{bmatrix}, \qquad (3.21)$$

[0174]   where S is the solution to the Lyapunov equation (3.9) and L_{n-1} is given by (3.12).

[0175]   To obtain the search direction, using Newton's method, we need the Hessian. Next, we describe how it is computed. Let the 2n×2n-matrix $\hat{P}$ be the solution to the Lyapunov equation

$$\hat{P} = \hat{A}'\hat{P}\hat{A} + \hat{c}'\hat{c},$$

[0176]   where $\hat{A}$ is the companion matrix (formed analogously to A in (3.10)) of the polynomial $\alpha(z)^2$ and $\hat{c}$ is the 2n row vector $(0, 0, \ldots, 0, 1)$. Analogously, determine the 3n×3n-matrix $\tilde{P}$ solving the Lyapunov equation

$$\tilde{P} = \tilde{A}'\tilde{P}\tilde{A} + \tilde{c}'\tilde{c},$$

[0177]   where $\tilde{A}$ is the companion matrix (formed analogously to A in (3.10)) of the polynomial $\alpha(z)^2\tau(z)$ and $\tilde{c}$ is the 3n row vector $(0, 0, \ldots, 0, 1)$. Then, the Hessian is

$$H = 2H_1 + H_2 + H_2' \qquad (3.22)$$

where

$$H_1 = L_n M(\rho) L(\alpha^2)^{-1} \begin{bmatrix} \hat{P} & 0 \\ 0 & 1 \end{bmatrix} L(\alpha^2)^{-1} M(\rho)' L_n \qquad (3.23)$$

$$H_2 = L_n M(\tau*\rho) L(\alpha^2\tau)^{-1} \begin{bmatrix} \tilde{P} & 0 \\ 0 & 1 \end{bmatrix} L(\alpha^2\tau)^{-1} M(\tau\rho)' \tilde{L}_n \qquad (3.24)$$

[0178]   where the precomputed matrices L_n and $\tilde{L}_n$ are given by (3.12) and by reversing the order of the rows in (3.12), respectively. Also $M(\rho)$, $M(\tau^*\rho)$ and $M(\tau\rho)$ are computed off-line, as in (3.20), whereas $L(\alpha^2)^{-1}$ and $L(\alpha\tau)^{-1}$ are computed in the following way:

[0179]   For an arbitrary polynomial (3.19), determine λ_0, $\lambda_1, \ldots, \lambda_m$ such that

$$\gamma(z)(\lambda_0 z^m + \lambda_1 z^{m-1} + \ldots + \lambda_m) = z^{2m} + \pi(z),$$

[0180]   where $\pi(z)$ is a polynomial of at most degree m−1. This yields m+1 linear equation for the m+1 unknowns λ_0, $\lambda_1, \ldots, \lambda_m$, from which we obtain

$$L(\gamma)^{-1} = \begin{bmatrix} \lambda_n & \cdots & \lambda_1 & \lambda_0 \\ \lambda_{n-1} & \cdots & \lambda_0 & \\ \vdots & \ddots & & \\ \lambda_0 & & & \end{bmatrix}.$$

[0181]   Finally, the new search direction becomes

$$d = H^{-1} \nabla J. \qquad (3.25)$$

[0182]   Let $d_{previous}$ denote the search direction d obtained in the previous iteration. If this is the first iteration, initialize by setting $d_{previous} = 0$

[0183]   Step 2. In this step a line search in the search direction d is performed. The basic elements are as follows. Five constants $c_j, j = 1,2,3,4,5$, are specified with suggested default values $C_1 = 10^{-10}$, $c_2 = 1.5$, $C_3 = 1.5$, $c_4 = 0.5$, and $c_5 = 0.001$. If this is the first iteration, set $\lambda = c_5$.

[0184]   If $\|d\| < c_2 \|d_{previous}\|$, increase the value of a parameter λ by a factor c_3. Otherwise, retain the previous value of λ. Using this λ, determine

$$h_{new} = h - \lambda d. \qquad (3.26)$$

[0185]   Then, an updated value for a is obtained by determining the polynomial (3.4) with all roots less than one in absolute value, satisfying

$$\alpha(z)\alpha(z^{-1}) = \sigma(z)\tau(z^{-1}) + \sigma(z^{-1})\tau(z)$$

[0186]   with $\sigma(z)$ being the updated polynomial (3.14) given by

$$\sigma(z) = \tau(z)q(z),$$

[0187]   where the updated q(z) is given by

$$q(z) = c(zI - A)^{-1}g + h_0, \quad g = \begin{bmatrix} h_n \\ \vdots \\ h_1 \end{bmatrix},$$

[0188]   with $h_n, h_{n-1}, \ldots, h_0$ being the components of h_{new}, A and C given by (3.10). This is a standard polynomial factorization problem for which there are several algorithms [F. L. Bauer, *Ein direktes Iterationsverfahren zur Hurwitz-Zerlegung eines Polynoms*, Arch. Elek. Ubertragung, 9 (1955), 285-290; Z. Vostrý, New algorithm for polynomial spectral factorization with quadratic convergence I, Kybernetika 77 (1975), 411-418], using only ordinary arithmetic operations. Hence they can be implemented with an MAC mathematics processing chip using ordinary skill in the art. However, the preferred method is described below (see explanation of routine q2a).

[0189]   This factorization can be performed if and only if q(z) satisfies condition (3.15). If this condition fails, this is determined in the factorization procedure, and then the value of λ is scaled down by a factor of c_4, and (3.26) is used to compute a new value for h_{new} and then of q(z) successfully until condition (3.15) is met.

[0190]   The algorithm is terminated when the approximation error given in (3.16) becomes less than a tolerance level specified by c_1, e.g., when

$$\sum_0^n (e_k)^2 < c_1.$$

[0191]   Otherwise, set h equal to h_{new} and return to Step 1.

[0192]   Description of technical steps in the procedure. The MATLAB code for this algorithm is given in Appendix B.

As an alternative a state-space implementation presented in C. I. Byrnes, T. T. Georgiou, and A. Lindquist, *A generalized entropy criterion for Nevanlinna-Pick interpolation: A convex optimization approach to certain problems in systems and control*, preprint, and C. I. Byrnes, T. T. Georgiou, and A. Lindquist, *A new approach to spectral estimation: A tunable high-resolution spectral estimator*, preprint, may also be used. The steps are conveniently organized in four routines:

[0193]   (1) Routine pm, which computes the Pick matrix from the given data $p = (p_0, p_1, \ldots, p_n)$ and $w = (w_0, w_1, \ldots, w_n)$

[0194]   (2) Routine q2a which is used to perform the technical step of factorization described in Step 2. More precisely, given q(z) we need to compute a rational function a(z) such that

$$a(z)a(z^{-1}) = q(z) + q(z^{-1})$$

[0195]   for the minimum-phase solution a(z), in terms of which $\alpha(z) = \tau(z)a(z)$. This is standard and is done by solving the algebraic Riccati equation

$$P - APA' - (g - APc')(2h_0 - cPc')^{-1}(g - APc') = 0,$$

[0196]   for the stabilizing solution. This yields

$$a(z) = c(zI - A)^{-1}(g - APc')/\sqrt{2h_0 - cPc'} + \sqrt{2h_0 - cPc'}.$$

[0197]   This is a standard MATLAB routine [W. F. Arnold, III and A. J. Laub, *Generalized Eigenproblem Algorithms and Software for Albebraic Riccati Equations*, Proc. IEEE, 72 (1984), 1746-1754].

[0198]   (3) Routine central, which computes the central solution as described above.

[0199]   (4) Routine decoder which integrates the above and provides the complete function for the decoder of the invention.

[0200]   An application to speaker recognition. In automatic speaker recognition a person's identity is determined from a voice sample. This class of problems come in two types, namely speaker verification and speaker identification. In speaker verification, the person to be identified claims an identity, by for example presenting a personal smart card, and then speaks into an apparatus that will confirm or deny this claim. In speaker identification, on the other hand, the person makes no claim about his identity, and the system must decide the identity of the speaker, individually or as part of a group of enrolled people, or decide whether to classify the person as unknown.

[0201]   Common for both applications is that each person to be identified must first enroll into the system. The enrollment (or training) is a procedure in which the person's voice is recorded and the characteristic features are extracted and stored. A feature set which is commonly used is the LPC coefficients for each frame of the speech signal, or some (nonlinear) transformation of these [Jayant M. Naik, *Speaker Verification: A tutorial*, IEEE Communications Magazine, January 1990, page 43], [Joseph P. Campbell Jr., *Speaker Recognition: A tutorial*, Proceedings of the IEEE 85 (1997), 1436-1462], [Sadaoki Furui, recent advances in Speaker Recognition, Lecture Notes in Computer Science 1206, 1997, page 239]. The motivation for using these is that

the vocal tract can be modeled using a LPC filter and that these coefficients are related to the anatomy of the speaker and are thus speaker specific. The LPC model assumes a vocal tract excited at a closed end, which is the situation only for voiced speech. Hence it is common that the feature selection only processes the voiced segments of the speech [Joseph P. Campbell Jr., *Speaker Recognition: A tutorial*, Proceedings of the IEEE 85 (1997), page 1455]. Since the THREE filter is more general, other segments can also be processed, thereby extracting more information about the speaker.

[0202] Speaker recognition can further be divided into text-dependent and text-independent methods. The distinction between these is that for text-dependent methods the same text or code words are spoken for enrollment and for recognition, whereas for text-independent methods the words spoken are not specified.

[0203] Depending on whether a text-dependent or text-independent method is used, the pattern matching, the procedure of comparing the sequence of feature vectors with the corresponding one from the enrollment, is performed in different ways. The procedures for performing the pattern matching for text-dependent methods can be classified into template models and stochastic models. In a template model as the Dynamic Time Warping (DTW) [Hiroaki Sakoe and Seibi Chiba, *Dynamic Programming Algorithm Optimization for Spoken Word Recognition*, IEEE Transactions on Acoustics, Speech and Signal Processing ASSP-26 (1978), 43-49] one assigns to each frame of speech to be tested a corresponding frame from the enrollment. In a stochastic model as the Hidden Markov Model (HMM) [L. R. Rabiner and B. H. Juang, *An Introduction to Hidden Markov Models*, IEEE ASSP Magazine, January 1986, 4-16] a stochastic model is formed from the enrollment data, and the frames are paired in such a way as to maximize the probability that the feature sequence is generated by this model.

[0204] For text-independent speaker recognition the procedure can be used in a similar manner for speech-recognition-based methods and text-prompted recognition [Sadaoki Furui, Recent advances in Speaker Recognition, Lecture Notes in Computer Science 1206, 1997, page 241f] where the phonemes can be identified.

[0205] Speaker verification. **FIG. 17** depicts an apparatus for enrollment. An enrollment session in which certain code words are spoken by a person later to be identified produces via this apparatus a list of speech frames and their corresponding MA parameters r and AR parameters a, and these triplets are stored, for example, on a smart card, together with the filter-bank parameters p used to produce them. Hence, the information encoded on the smart card (or equivalent) is speaker specific. When the identity of the person in question needs to be verified, the person inserts his smart card in a card reader and speaks the code words into an apparatus as depicted in **FIG. 18**. Here, in Box **12**, each frame of the speech is identified. This is done by any of the pattern matching methods mentioned above. These are standard procedures known in the literature [Joseph P. Campbell Jr., *Speaker Recognition: A tutorial*, Proceedings of the IEEE 85 (1997), pages 1452-1454]. From the smart card the corresponding r, a and p are retrieved. The filter-bank poles are transferred to the Bank of Filters and the Decoder. (Another p could be used, but the same has to be used in both

Box **1** and Box **7**.) The parameters r and a are also transferred to the Decoder. The AR parameters a are used as initial condition in the Decoder algorithm (unless the central solution is used in which case no initial condition is needed). Box **7** produces AR parameters â which hopefully are close to a. The error â-a from each frame is compounded in a measure of goodness-of-fit, and decision is finally made as to whether to accept or reject the person.

[0206] Speaker identification. In speaker identification the enrollment is carried out in a similar fashion as for speaker verification except that the feature triplets are stored in a database. **FIG. 19** depicts an apparatus for speaker identification. It works like that in **FIG. 17** except that there is a frame identification box (Box **12**) as in **FIG. 18**, the output of which together with the MA parameters a and AR parameters a are fed into a data base. The feature triplets are compared to the corresponding triplets for the population of the database and a matching score is given to each. On the basis of the (weighted) sum of the matching scores of each frame the identity of the speaker is decided.

[0207] Doppler-Based Applications and Measurement of Time-Delays. In communications, radar, sonar and geophysical seismology a signal to be estimated or reconstructed can often be described as a sum of harmonics in additive noise [P. Stoica and Ro. Moses, *Introduction to Spectral Analysis*, Prentice-Hall, 1997, page 139]. While traditional methods are designed for either white noise or no noise at all, estimation of sinusoids in colored noise has been regarded as difficult problem [B. Porat, *Digital Processing of Random Signals*, Prentice-Hall, 1994, pages 285-286]. THREE filter design is particularly suited for the colored noise case, and as an ARMA method it offers "super-resolution" capabilities [P. Stoica and Ro. Moses, *Introduction to Spectral Analysis*, Prentice-Hall, 1997, page 136]. As an illustration, see the second example in the introduction.

[0208] Tunable high-resolution speed estimation by Doppler radar. We disclose an apparatus based on THREE filter design for determining the velocities of several moving objects. If we track m targets moving with constant radial velocities $v_1, v_2, \ldots, v_m$, respectively, by a pulse-Doppler radar emitting a signal of wave-length λ, the backscattered signal measured by the radar system after reflection of the objects takes the form

$$y(t) = \sum_{k=1}^{m} \alpha_k e^{i\theta_k t} + v(t),$$

[0209] where $\theta_1, \theta_2, \ldots, \theta_m$ are the Doppler frequencies, $v(t)$ is the measurement noise, and $\alpha_1, \alpha_2, \ldots, \alpha_m$ are (complex) amplitudes. (See, e.g., B. Porat, *Digital Processing of Random Signals*, Prentice-Hall, 1994, page 402] or [P. Stoica and Ro. Moses, *Introduction to Spectral Analysis*, Prentice-Hall, 1997, page 175].) The velocities can then be determined as

$$v_k = \frac{\lambda \theta_k}{4\pi\Delta} \quad k = 1, 2, \ldots, m,$$

[0210] where Δ is the pulse repetition interval, assuming once-per-pulse coherent in-phase/quadrature sampling.

14

[0211] **FIG. 20** illustrates a Doppler radar environment for our method, which is based on the Encoder and Spectral Analyzer components of the THREE filter. To estimate the velocities amounts to estimating the Doppler frequencies which appear as spikes in the estimated spectrum, as illustrated in **FIG. 7**. The device is tuned to give high resolution in the particular frequency band where the Doppler frequencies are expected.

[0212] The only variation in combining the previously disclosed Encoder and Spectral Estimator lies in the use of dashed rather than solid communication links in **FIG. 20**. The dashed communication links are optional. When no sequence r of MA parameters is transmitted from Box **6** to Box **7'**, Box **7'** chooses the default values $r=(\tau_1, \tau_2, \ldots, \tau_m)$, which are defined via (3.1) in terms of the sequence p of filter-bank parameters, transmitted by Component **4** to Box **7'**. In the default case, Box **7'** also transmits the default values $r=\tau$ to Box **10**. For those applications when it is desirable to tune the MA parameters sequence r from the observed data stream, as disclosed above, the dotted lines can be replaced by solid (open) communication links, which then transmit the tuned values of the MA parameter sequence r from Box **6** to Box **7'** and Box **10**.

[0213] The same device can also be used for certain spatial doppler-based applications [P. Stoica and Ro. Moses, *Introduction to Spectral Analysis*, Prentice-Hall, 1997, page 248].

[0214] Tunable high-resolution time-delay estimator. The use of THREE filter design in line spectra estimation also applies to time delay estimation [M. A. Hasan and M. R. Azimi-Sadjadi, *Separation of multiple time delays using new spectral estimation schemes*, IEEE Transactions on Signal Processing 46 (1998), 2618-2630][M. Zeytinoglu and K. M. Wong, *Detection of harmonic sets*, IEEE Transactions on Signal Processing 43 (1995), 2618-2630] in communication. Indeed, the tunable resolution of THREE filters can be applied to sonar signal analysis, for example the identification of time-delays in underwater acoustics [M. A. Hasan and M. R. Azimi-Sadjadi, *Separation of multiple time delays using new spectral estimation schemes*, IEEE Transactions on Signal Processing 46 (1998), 2618-2630].

[0215] **FIG. 21** illustrates a possible time-delay estimator environment for our method, which has precisely the same THREE-filter structure as in **FIG. 20** except for the preprocessing of the signal. In fact, this adaptation of THREE filter design is a consequence of Fourier analysis, which gives a method of interchanging frequency and time. In more detail, if x(t) is the emitted signal, the backscattered signal is of the form

$$z(t) = \sum_{k=1}^{m} h_k(t) * x(t - \delta_k) + v(t),$$

[0216] where the first term is a sum of convolutions of delayed copies of the emitted signal and v(t) represents ambient noise and measurement noise. The convolution kernels h_k, k=1, 2, . . . , m, represent effects of media or reverberation [M. A. Hasan and M. R. Azimi-Sadjadi, *Separation of multiple time delays using new spectral estimation schemes*, IEEE Transactions on Signal Processing 46 (1998), 2618-2630], but they could also be δ-functions with

Fourier transforms $H_k(\omega) \equiv 1$. Taking the Fourier transform, the signal becomes

$$Z(\omega) = \sum_{k=1}^{m} H_k(\omega) X(\omega) e^{i\omega \delta_k} + n(\omega),$$

[0217] where the Fourier transform $X(\omega)$ of the original signal is known and can be divided off.

[0218] It is standard in the art to obtain a frequency-dependent signal from the time-dependent signal by fast Fourier methods, e.g., FFT. Sampling the signal Z(w) at frequencies $\omega = \tau \omega_0$, $\tau = 0$, 1, 2, . . . , N, and using our knowledge of the power spectrum $X(\omega)$ of the emitted signal, we obtain an observation record

[0219] $y_0, y_1, y_2, \ldots, y_N$

[0220] of a time series

$$y(\tau) = \sum_{k=1}^{m} a_k e^{i\tau \theta_k} + v(\tau),$$

[0221] where $\theta_k = \omega_0 \delta_k$ and $v(\tau)$ is the corresponding noise. To estimate spectral lines for this observation record is to estimate θ_k, and hence δ_k for k=1, 2, . . . , m. The method and apparatus described in **FIG. 20** is then a THREE line-spectra estimator as the one disclosed above and described in **FIG. 20** with the modifications described here. In particular, the Transmitter/Receiver could be a sonar.

[0222] Other Areas of Application. The THREE filter method and apparatus can be used in the encoding and decoding of signals more broadly in applications of digital signal processing. In addition to speaker identification and verification, THREE filter design could be used as a part of any system for speech compression and speech processing. The use of THREE filter design line spectra estimation also applies to detection of harmonic sets [M. Zeytinoglu and K. M. Wong, *Detection of harmonic sets*, IEEE Transactions on Signal Processing 43 (1995), 2618-2630]. Other areas of potential importance include identification of formants in speech and data decimation [M. A. Hasan and M. R. Azimi-Sadjadi, *Separation of multiple time delays using new spectral estimation schemes*, IEEE Transactions on Signal Processing 46 (1998), 2618-2630]. Finally, we disclose that the fixed-mode THREE filter, where the values of the MA parameters are set at the default values determined by the filter-bank poles also possesses a security feature because of its fixed-mode feature: If both the sender and receiver share a prearranged set of filter-bank parameters, then to encode, transmit and decode a signal one need only encode and transmit the parameters w generated by the bank of filters. Even in a public domain broadcast, one would need knowledge of the filter-bank poles to recover the transmitted signal.

[0223] Various changes may be made to the invention as would be apparent to those skilled in the art. However, the invention is limited only to the scope of the claims appended hereto, and their equivalents.

APPENDIX A

Determination of Spectral Zeros

There are several alternatives for tuning the MA parameters (2.4). First, using the Autocorrelation Method [T.P. Barnwell III, K. Nayebi and C.H. Richardson, *Speech Coding: A Computer Laboratory Textbook*, John Wiley & Sons, New York, 1996, pages 91-93], or some version of Burg's algorithm [B. Porat, *Digital Processing of Random Signals*, Prentice Hall, 1994, page 176], we first compute the PARCOR coefficients (also called reflection coefficients)

$$\gamma_0, \gamma_1, \cdots, \gamma_{m+n}$$

for some $m \geq n$, and then we solve the Toeplitz system

$$\begin{bmatrix} \gamma_m & \gamma_{m-1} & \cdots & \gamma_{m+1-n} \\ \gamma_{m+1} & \gamma m & \cdots & \gamma_{m+2-n} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{m+n-1} & \gamma_{m+n-2} & \cdots & \gamma_m \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} = - \begin{bmatrix} \gamma_{m+1} \\ \gamma_{m+2} \\ \vdots \\ \gamma_{m+n} \end{bmatrix} \qquad (A.1)$$

for the parameters $r_1, r_2, \cdots, r_n$. If the polynomial

$$\rho(z) = z^n + r_1 z^{n-1} + \cdots + r_n,$$

has all its roots less than one in absolute value, we use $r_1, r_2, \cdots, r_n$ as MA parameters. If not, we take $\rho(z)$ to be the stable spectral factor of $\rho(z)\rho(z^{-1})$, obtained by any of the factorization algorithms in Step 2 in the Decoder algorithm, and normalized so that the leading coefficient (that of z^n) is 1.

Alternative methods can be based on any of the procedures described in [J.D. Markel and A.H. Gray, *Linear Prediction of Speech*, Springer Verlag, Berlin, 1976, pages 271-275], including Prony's method with constant term. These methods are not by themselves good for producing, for example, synthetic speech, because they do not satisfy the interpolation conditions. However, here we use only the zero computation, the corresponding poles being determined by our methods. Alternatively, the zeros can also be chosen by determining the phase and the moduli of the zeros from the notches in an observed spectrum, as represented by a periodogram or as computed using Fast Fourier Transforms (FFT). This is depicted in Figure 22 where a
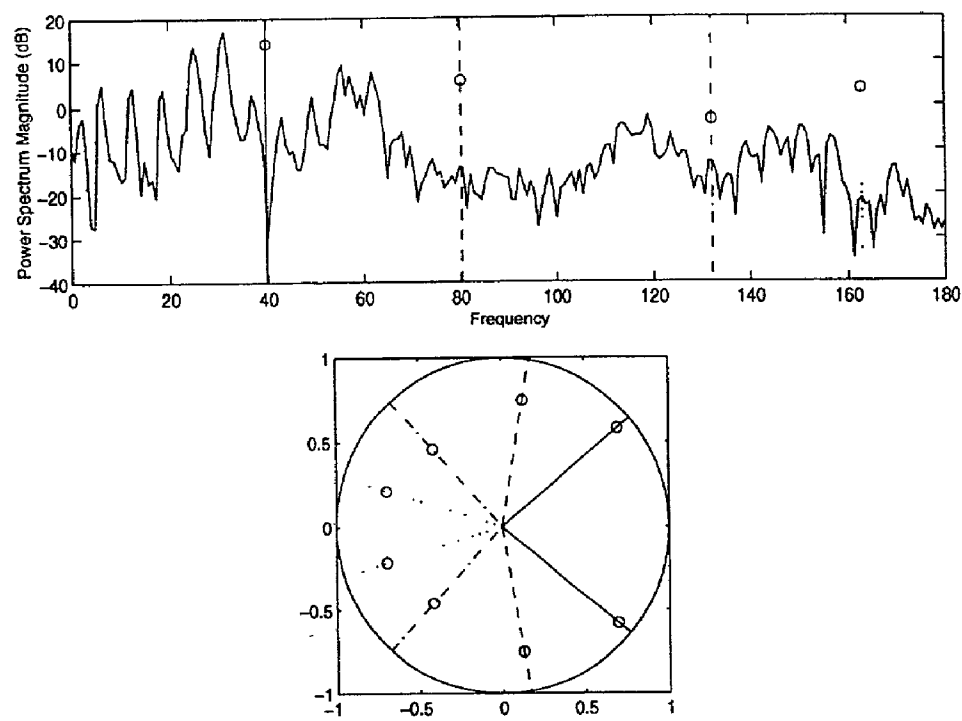
51383464.doc

Figure 22: Invention: Selecting the zeros from a periodogram.

APPENDIX B

Routine PM

```
function Pick=pm(p,w)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%
%%     function Pick=pm(p,w,option)
%%
%% Works with scalar data
%%
%%         |p|<1,    w in C
%%
%% It corresponds to:
%%
%%         p --> w
%%
%% Computes the Pick matrix for the corresponding Caratheodory problem
%%
%%         Pick=[(wk+wj^*)/(1-pk*pj^*)]_(k,j)
%%
%% (the Pick matrix is unitarily equivalent to the one corresponding
%%  to the assignment p^(-1) --> w.)
%%
%% NOTE:       p's must not be repeated
%%
%% DEFAULT:  If p_i is contain in p, the so is conj(p_i).
%%
%%             If the conjugate values ARE NOT part of the array,
%%             then set OPTION to any nonzero value.
%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
if length(p)~=length(w), disp('p,w ought to have the same size'), return, end
p=p(:); w=w(:); n=length(p);

if nargin==3,
 for i=1:n,
  if abs(imag(p(i)))>10*eps, p=[p; conj(p(i))]; w=[w; conj(w(i))];
  elseif  abs(imag(w(i)))>10*eps, disp(' real p_i should correspond with real w_i'),
                                disp('                        -- TERMINATED'),
  else, p(i)=real(p(i)); w(i)=real(w(i));
  end,
 end,
end
```

```
n=length(p);
Pick=((w*ones(1,n)+ones(n,1)*w')./(ones(n,n)-p*p'));


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Last line of pm.m (August 30, 1998)
```

ROUTINE Q2A

```
function [a,flag]=q2a(tau,hq)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%
%%       function [a,flag]=q2a(tau,h)
%%
%% Contstructs a function a(z) from q(z) such that
%%
%%               q+q^* = a a^*
%%
%% NOTE: q(z) must be strictly positive real.
%%
%% Representation convention: e.g., q=[compan(tau) bq; (0 ...0 1); dq]
%% and h=flipud([bq;dq])' is the row vector of Markov parameters.
%%
%% Needs
%%       control toolbox:   DARE (sets parameter FLAG)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
hq=flipud(hq(:));
dq=hq(1);
bq=hq; bq(1)=[]; bq=flipud(bq);
   if bq==0,
      if dq<0, flag=1; a=[]; return, end
      flag=0;
   da=sqrt(2*dq); ba=bq; ha=[da; flipud(ba)].';
%%% hd2n
tau=tau(:).';
ha=flipud(ha(:)).';
a=conv(tau,ha);
a=a(1:length(tau));
   return,
   end
```

```
A=compan(tau); c=[zeros(1,length(tau)-2) 1];
%%% SOLVING
% P-A*P*A'- (bq - A*P*c')*(dq+dq' - c*P*c')^-1 *(bq - A*P*c')'
 [P,L,G,flag]=dare(A',c',zeros(size(A)),-dq-dq',-bq,'report');
% [X1,X2,L,flag]=dare(A',c',zeros(size(A)),-dq-dq',-bq,'implicit');
if abs(flag)>1e-5, a=[]; return, end

da=sqrt(dq+dq'-c*P*c');
ba=(bq-A*P*c')*inv(da');
ha=[ba; da];

%%% hd2n
tau=tau(:).';
ha=flipud(ha(:)).';
a=conv(tau,ha);
a=a(1:length(tau));
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Last line of q2a.m (September 5, 1998).
```

ROUTINE CENTRAL

```
function [num,den]=central(p,w)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%
%% function [num,den]=central(p,w)
%%
%% Computes the central solution corresponding to interpolation data
%% Z=[p^{-1}(1); p^{-1}(2); ... p^{-1}(n)] and w=[w(1); w(2); ... w(n)], where
%%
%%    f(p(i)^{-1}) = w(i)   NOTE: p's are inside the unit disc
%%                          and interpolation is required at their reflection as well
%%                          with the corresponding conjugate value.
%%
%% The solution is provided in either
%%      (i)  the form of a positive real function
%%
%%               f(z)=num(z)/den(z)
%%
%% It assumes (1) that p(1) be 0; this a convenient normalization,
%%            (2) that if pk is in p then pk^* is NOT in p, <<<<<<<<< IMPORTANT
%%
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%% Uses Matlab built-in: ss2tf, and tf2ss for transforming between
%% state-space [a,b,c,d] and transfer [num,den] representations.
%% This is standard and in the absence of ss2tf, tf2ss, one can use instead:
%%                          -
% function [a,b,c,d]=tf_to_ss(num,den)
% a=compan(den);
% c=fliplr(eye(1,length(a)));
% R=hankel(flipud(eye(length(den),1)),den);
% bd=inv(R)*[zeros(1,length(R)-length(num)) num]';
% b=bd; b(length(b))=[];
% d=bd(length(bd));
% return
%%
%%
% function [num,den]=ss_to_tf(a,b,c,d)
% den=poly(a);
% h=d;
% for k=1:length(a),
% h=[c*a^(k-1)*b; h];
% end
% R=hankel(flipud(eye(length(den),1)),den);
% num=R*h; num=num(:).';
% return
%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Pick=pm(p,w,1);  if min(real(eig(Pick)))<=0,
                 disp('The Pick matrix is not positive'), return,
                 end

if abs(p(1))<10*eps, p(1)=0; w(1)=real(w(1)); w0=w(1);
                 else, disp('p(1) is required to be 0'), return, end

p=p(:); w=w(:)/w0;
pn=p; pn(1)=[];
wn=w; wn(1)=[];
n_temp=length(pn);
pnc=pn;
for i=1:n_temp,
                 if abs(imag(pn(i)))>10*eps, pnc=[pnc; conj(pn(i))];
                 else, pn(i)=real(pnc(i));   pnc(i)=pn(i);
                 end,
end,
```

```
n=length(pnc);
tau=poly(pnc);

vn=(ones(size(wn))-wn)./(ones(size(wn))+wn);
sn=(ones(size(pn))-pn)./(ones(size(pn))+pn);

snc=(ones(size(pnc))-pnc)./(ones(size(pnc))+pnc);
tauhat=poly(-snc);

un=vn.*polyval(tauhat,sn);
ur=real(un); ui=imag(un);

U=[];
  for i=1:n, U=[sn.^(i-1) U]; end,
Ur=real(U); Ui=imag(U);
  for i=n_temp:-1:1,
    if ((abs(Ui(i,:))<1e2*eps) & (abs(ui(i,:))<1e2*eps)), Ui(i,:)=[]; ui(i,:)=[]; en
  end

UU=[Ur;Ui]; uu=[ur;ui];
pihat=UU\uu; pihat=pihat(:).';
[a,b,c,d]=tf2ss(pihat,tauhat);

Po=lyap(a',-c'*c);
A=a-inv(Po)*c'*c;
Lc=lyap(A,-b*b');
N=inv(eye(size(Lc))+Po*Lc);

[N1,D1]=ss2tf(a,inv(Po)*N*c',  c,1);
[N2,D2]=ss2tf(a,N'*b,          c,0);
[N3,D3]=ss2tf(a,inv(Po)*N*c', -b'*Po,0);
[N4,D4]=ss2tf(a,N'*b,         -b'*Po,1);

%%%%%%%%%%%%%%%%%%% C2D transformation matrix (denoted T' in disclosure) %%%%
%%%%%%%%%%%%%%%%%%%    numd/dend=c2d(numc/denc,n), by setting s=(z-1)/(z+1)
C2D=[];
for i=0:n,
  row=1;
  for j=i+1:n, row=conv(row,[1 -1]); end
  for k=1:i,   row=conv(row,[1 +1]); end
  C2D=[C2D;row];
end, clear row

num1=N1*C2D;
```

```
num2=N2*C2D;
num3=N3*C2D;
num4=N4*C2D;

mu=-num2(1)/num1(1);

ahat=(mu*(num3+num1)+(num4+num2))/sqrt(1-mu^2);
bhat=w0*(mu*(num3-num1)+(num4-num2))/sqrt(1-mu^2);

den=sum(tau.^2)/(2*ahat*bhat')*ahat;
num=sum(tau.^2)/(2*ahat*bhat')*bhat;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Last line of central.m (September 24, 1998)
```

ROUTINE DECODER

```
function [b,a] = decoder(p,w,r,Init,lambda)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%
%%     function [b,a] = decoder(p,w,r,Init,lambda)
%%
%% Standing Assumptions: (i)  p_0=0,
%%                       (ii) If p_i is complex conj(p_i), then is not in p.
%%
%% Init=1 or 2 (choice of initialization 1 or 2)
%% lambda = initial choice for correction scaling in updating h --> h - lambda*d
%%          (default: lambda=1e-3, dynamically adjusted in subsequent steps).
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% NEEDS: pm.m, central.m
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

c1=1e-10;
c2=1.5;
c3=1.5;
c4=.5;

%%%%%%%%%%%%%% BEGINNING CHECKS and SETTING UP DATA (p,w,n,nc,tau,r) %%%%%%%%%%
%%
if       nargin<2, disp('Less than 2 arguments       -- TERMINATED'), return, end
p=p(:); w=w(:); n=length(p)-1;
```

```
pinitial=p; winitial=w;
if length(w)~=n+1,        disp('length(p)~=length(w)  -- TERMINATED'), return, end

if abs(p(1))>10*eps, disp('p(1) ought to be 0        -- TERMINATED'), return, end
p(1)=0; w(1)=real(w(1));
for i=2:n+1,
  if abs(imag(p(i)))>10*eps, p=[p; conj(p(i))]; w=[w; conj(w(i))];
  elseif abs(imag(w(i)))>1000*eps, disp(' real p_i should correspond with real w_i')
                                   disp('         -- TERMINATED'), return,
  else, p(i)=real(p(i)); w(i)=real(w(i));
  end,
end,

nc=length(p)-1;
tau=poly(p); tau(nc+2)=[];

if         nargin<3, r=tau; r(1)=[]; Init=1; lambda=1e-3;
   elseif nargin<4, r=r(:).'; Init=1; lambda=1e-3;
   elseif nargin<5, lambda=1e-3; end

if length(r)>nc,          disp('length(r)>length(w)-1 -- TERMINATED'), return,
   else, r=[r zeros(1,nc-length(r))]; end

Pick=pm(p,w);
if min(real(eig(Pick)))<=0, disp('The Pick matrix is not positive'),
   eigPickEinv=eig( Pick/pm(p,ones(size(p))) );
   level=min(real(eigPickEinv));
   disp('ATTENTION: w_s will be raised by '), rise=-level+1e6*eps,
   w=w+rise*ones(size(w));
   end
clear Pick eigPickEinv level rise

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%% Central solution
if r==tau(2:nc+1), [b,a]=central(p(1:n+1),w(1:n+1)); return, end

%%%%%%%%%%%%% STARTING COMPUTATIONS (off line computed once) %%%%%%%%%%%%%%%%%
%%%%%%%%%%%%% STEP 0 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%
A=compan(tau); c=fliplr(eye(1,nc));
L=hankel(flipud(eye(nc+1,1)),tau);
S=dlyap(A',c'*c);                                %%% Solving: S-A'*S*A-c'*c=0
tau_1=tau; tau_1(nc+1)=[];
L_1=hankel(flipud(eye(nc,1)),tau_1);
```

```
SLinv=S*inv(L_1);

%%% INITIALIZATION 1 (default) %%%%%%%%%%%%%%%
if Init==1,
y=L\[1;r(:)];                                             %%% Also: y=nd2h([1 r],tau)
kappa=y'*[S zeros(nc,1); zeros(1,nc) 1]*y;
a=sqrt(kappa/2/w(1))*tau;

%%% INITIALIZATION 2 %%%%%%%%%%%%%%%%%%%%%%%%
else,
[numf,a]=central(p(1:n+1),w(1:n+1)); a=a*abs(1+sum(r))/abs(sum(tau));
end,

%%% Algorithm (DATA) %%%%%%%%%%%%%%%%%%%%%%%%
%%% h: markov parameters of initial q %%%%%%%
a=[a zeros(1,length(tau)-length(a))];
nq=(a*hankel(a))/( hankel(tau)+toeplitz(tau,tau(1)*eye(1,length(tau))) );

%%% nd2h
nq=nq(:).';
nq=[zeros(1,length(tau)-length(nq)) nq];
h=deconv([nq zeros(1,length(tau)-1)],tau);
h=flipud(h(:));

%%% pis=tau rhostar + rho taustar %%%%%%%%%%%%
pis=(toeplitz(eye(nc+1,1),[1 r])*[1;r(:)]).';

%%% Vandermonde matrix %%%%%%%%%%%%%%%%%%%%%%%
z12n=p(2:n+1).^-1;
V=ones(n,1); for i=1:nc-1, V=[z12n.^i V]; end,
V_ri=[real(V);imag(V)];
tau_z12n=polyval(tau,z12n);

%%% M(gamma) %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
g=[1 r];
M_rho      = toeplitz(eye(nc+1,1)*g(1), [g zeros(1,nc)]);

g=conv(fliplr(tau),[1 r]);
M_taus_rho= toeplitz(eye(nc+1,1)*g(1), [g zeros(1,nc)]);

g=conv(tau,[1 r]);
M_tau_rho = toeplitz(eye(nc+1,1)*g(1), [g zeros(1,nc)]);

clear Init S kappa numf s g nn V L_1 tau_1
```

```
%%%%%%%%%%%% ITERATION %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% AVAILABLE a,h  (a: coefficients of alpha, h: markov parameters of q)

b = ( hankel(a)+fliplr(hankel(fliplr(a))) )\pis(:); b=b.';
  e0=w(1)-b(1)/a(1);
  e=w(2:n+1)-polyval(b,z12n)./polyval(a,z12n);
  approximation_error=norm([e0;e]);

d_past=0;
desired_minimal_error=c1*norm(w);
while approximation_error>desired_minimal_error,

%%%%%%%%%%%% STEP 1 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
  v=(e-e0*ones(size(e))).*tau_z12n;
  v_ri=[real(v);imag(v)];
  x=V_ri\v_ri;

grad=2*[SLinv*x;2*e0];
                        clear x v_ri v e e0 b

  a_2=conv(a,a);          gamma_2=deconv(eye(1,2*length(a_2)-1),a_2);
  a_2_tau=conv(a_2,tau);  gamma_2_tau=deconv(eye(1,2*length(a_2_tau)-1),a_2_tau);

  L_a2inv=hankel(fliplr(gamma_2));
  L_a2tauinv=hankel(fliplr(gamma_2_tau));

  Phat = dlyap( compan(a_2).'   , [zeros(2*nc-1,2*nc); fliplr(eye(1,2*nc))]);
  Ptilde=dlyap( compan(a_2_tau).', [zeros(3*nc-1,3*nc); fliplr(eye(1,3*nc))]);

H1=L*M_rho*L_a2inv*[Phat zeros(2*nc,1);zeros(1,2*nc) 1]*L_a2inv*M_rho.'*L;
H2=L*M_taus_rho*L_a2tauinv*[Ptilde zeros(3*nc,1);zeros(1,3*nc) 1]*L_a2tauinv*M_tau_rh
H=2*H1+H2+H2.';

d    =H\grad;
%%%%%%%%%%%% STEP 2,3 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%%%%%%%%%%%% accelerating criterion (increasing "lambda")
if norm(d_past)<norm(d)*c2, lambda=min(lambda*c3,1); end
d_past=d;

%%%%%%%%%%%% step possibly decreasing "lambda"
flag=1;      lambda=lambda/c4;
while abs(flag)>1e-5,
```

```
            lambda=lambda*c4;
hnew=h-lambda*d;
[a,flag]=q2a(tau,hnew);
end

h=hnew;

b = ( hankel(a)+fliplr(hankel(fliplr(a))) )\pis(:); b=b.';
  e0=w(1)-b(1)/a(1);
  e=w(2:n+1)-polyval(b,z12n)./polyval(a,z12n);
  approximation_error=norm([e0;e]);
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Last line of decoder.m (September 21, 1998)
```

APPENDIX C

A CONVEX OPTIMIZATION APPROACH TO THE RATIONAL COVARIANCE EXTENSION PROBLEM*

CHRISTOPHER I. BYRNES†, SERGEI V. GUSEV‡, AND ANDERS LINDQUIST§

Abstract. In this paper we present a convex optimization problem for solving the rational covariance extension problem. Given a partial covariance sequence and the desired zeros of the modeling filter, the poles are uniquely determined from the unique minimum of the corresponding optimization problem. In this way we obtain an algorithm for solving the covariance extension problem, as well as a constructive proof of Georgiou's seminal existence result and his conjecture, a stronger version of which we have resolved in [7].

Key words. rational covariance extension, partial stochastic realization, trigonometric moment problem, spectral estimation, speech processing, stochastic modeling

AMS subject classifications. 30E05, 60G35, 62M15, 93A30, 93E12

1. Introduction

In [7] a solution to the problem of parameterizing all rational extensions of a given window of covariance data has been given. This problem has a long history, with antecedents going back to potential theory in the work of Carathéodory, Toeplitz and Schur [9, 10, 31, 30], and continuing in the work of Kalman, Georgiou, Kimura and others [18, 14, 21]. It has been of more recent interest due to its significant interface with problems of in signal processing and speech processing [11, 8, 25, 20] and in stochastic realization theory and system identification [2, 32, 22]. Indeed, the recent solution to this problem, which extended a result by Georgiou and answered a conjecture by him [13, 14] in the affirmative, has shed some light on the stochastic (partial) realization problem through the development of an associated Riccati-type equation, whose unique positive semi-definite solution has as its rank the minimum dimension of a stochastic linear realization of the given rational covariance extension [6]. In both its form as a complete parameterization of rational extensions to a given covariance sequence and as an indefinite Riccati-type equation, one of the principal problems which remains open is that of developing effective computational methods for the approximate solution of this problem. In this paper, motivated by the effectiveness of interior point methods for solving nonlinear convex optimization problems, we recast the fundamental problem as such an optimization problem.

In Section 2 we describe the principal results about the rational covariance extension problem, while setting notation we shall need throughout. The only solution to this problem for which there has been simple computational procedures is the so called *maximum entropy* solution, which is the particular solution that maximizes the entropy gain. In Section 3 we demonstrate that the infinite-dimensional optimization problem for determining this solution has a simple finite-dimensional dual. This motivates the introduction in Section 4 of a nonlinear, strictly convex functional defined on a closed convex set naturally related to the covariance extension problem. We first show that any solution of the rational covariance extension problem lies in the interior of this convex set and that, conversely, an interior minimum of this convex functional will correspond to the unique solution of the covariance extension problem. Our interest in this convex optimization problem is, therefore, twofold: as a starting point for the computation of an explicit solution, and as a means of providing an alternative proof of the rational covariance extension theorem.

Concerning the existence of a minimum, we show that this functional is proper and bounded below, i.e., that the sublevel sets of this functional are compact. From this, it follows that there exists a minimum. Since uniqueness follows from strict convexity of the functional, the central issue which needs to be addressed in order to solve the rational covariance extension problem is whether, in fact, this minimum is an interior point. Indeed, our formulation of the convex functional, which contains a barrier-like term, was inspired by interior point methods. However, in contrast to interior point methods, the barrier function we have introduced does not become infinite on the boundary of our closed convex set. Nonetheless, we are able to show that the gradient, rather than the value, of the convex functional becomes infinite on the boundary. The existence of an interior point which minimizes the functional then follows from this observation.

In Section 5, we apply these convex minimization techniques to the rational covariance extension problem, noting that, as hinted above, we obtain a new proof of Georgiou's conjecture. Moreover, this proof, unlike our previous proof [7] and the existence proof of Georgiou [14], is constructive. Consequently, we have also obtained an algorithmic procedure for solving the rational covariance extension problem. In Section 6 we report some computational results and present some simulations.

2. The rational covariance extension problem

It is well-known that the spectral density $\Phi(z)$ of a purely nondeterministic stationary random process $\{y(t)\}$ is given by the Fourier expansion

$$\Phi(e^{i\theta}) = \sum_{-\infty}^{\infty} c_k e^{ik\theta} \qquad (2.1)$$

on the unit circle, where the covariance lags

$$c_k = E\{y_{t+k} y_t\} \quad k = 0, 1, 2, \ldots \qquad (2.2)$$

play the role of the Fourier coefficients

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\theta} \Phi(e^{i\theta}) d\theta. \qquad (2.3)$$

CONVEX OPTIMIZATION FOR RATIONAL COVARIANCE EXTENSIONS   3

In spectral estimation [8], identification [2, 22, 32], speech processing [11, 25, 24, 29] and several other applications in signal processing and systems and control, one is faced with the inverse problem of finding a spectral density, which is *coercive*, i.e., positive on the unit circle, given only

$$c = (c_0, c_1, \ldots, c_n),$$ (2.4)

which is a *partial covariance sequence* positive in the sense that

$$T_n = \begin{bmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_0 & \cdots & c_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n-1} & \cdots & c_0 \end{bmatrix} > 0,$$ (2.5)

i.e., the Toeplitz matrix T_n is positive definite.

In fact, the covariance lags (2.2) are usually estimated from an approximation

$$\frac{1}{N-k+1} \sum_{t=0}^{N-k} y_{t+k} y_t$$

of the ergodic limit

$$c_k = \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T} y_{t+k} y_t,$$

since only a finite string

$$y_0, y_1, y_2, y_3, \ldots, y_N$$

of observations of the process $\{y(t)\}$ is available, and therefore we can only estimate a finite partial covariance (2.4) where $n \ll N$.

The corresponding inverse problem is the left with a version of the *trigonometric moment problem*: Given a sequence (2.4) of real numbers satisfying the positivity condition (2.5), find a coercive spectral density $\Phi(z)$ such that (2.3) is satisfied for $k = 0, 1, 2, \ldots, n$. Of course there are infinitely many such solutions, and we shall shortly specify some additional properties which we would like the solution to have.

The trigonometric moment problem, as stated above, is equivalent to the *Carathéodory extension problem* to determine an extension

$$c_{n+1}, c_{n+2}, c_{n+3}, \ldots$$ (2.6)

with the property that the function

$$v(z) = \frac{1}{2} c_0 + c_1 z^{-1} + c_2 z^{-2} + \ldots$$ (2.7)

is *strictly positive real*, i.e., is analytic on and outside the unit circle (so that the Laurent expansion (2.7) holds for all $|z| \geq 1$) and satisfies

$$v(z) + v(z^{-1}) > 0 \quad \text{on the unit circle.}$$ (2.8)

In fact, given such a $v(z)$,

$$\Phi(z) = v(z) + v(z^{-1})$$ (2.9)

is a solution to the trigonometric moment problem. Conversely, any coercive spectral density $\Phi(z)$ uniquely defines a strictly positive real function $v(z)$ via (2.9).

4   C. I. BYRNES, S. V. GUSEV, AND A. LINDQUIST

These problems are classical and go back to Carathéodory [9, 10], Toeplitz [31] and Schur [30]. In fact, Schur parameterized all solutions in terms of what is now known as the *Schur parameters*, or, which is more common in the circuits and systems literature, *reflection coefficients*, and which are easily determined from the covariance lags via the Levinson algorithm [27]. More precisely, modulo the choice of c_0, there is a one-to-one correspondence between infinite covariance sequences $c_0, c_1, c_2, \ldots$ and Schur parameters $\gamma_0, \gamma_1, \ldots$ such that

$$|\gamma_t| < 1 \quad \text{for } t=0,1,2, \ldots \tag{2.10}$$

under which partial sequences (2.4) corresponds to partial sequences $\gamma_0, \gamma_1, \ldots, \gamma_{n-1}$ of Schur parameters. Therefore, covariance extension (2.6) amounts precisely to finding a continuation

$$\gamma_n, \gamma_{n+1}, \gamma_{n+2}, \ldots \tag{2.11}$$

of Schur parameters satisfying (2.10). Each such solution is only guaranteed to yield a $v(z)$ which is meromorphic.

In circuits and systems theory, however, one is generally only interested in solutions which yield a rational $v(z)$ of at most degree n, or, which is equivalent, a rational spectral density $\Phi(z)$ of at most degree $2n$. Then, the unique rational, stable, minimum-phase function $w(z)$ having same degree as $v(z)$ and satisfying

$$w(z)w(z^{-1}) = \Phi(z) \tag{2.12}$$

is the transfer function of a *modeling filter*, which shapes white noise into a random process with the first $n+1$ covariance lags given by (2.4); see e.g. [7, 6] for more details.

Setting all free Schur parameters (2.11) equal to zero, which clearly satisfies the condition (2.10), yields a rational solution

$$\Phi(z) = \frac{1}{a(z)a(z^{-1})}, \tag{2.13}$$

where $a(z)$ is a polynomial

$$a(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n \quad (a_0 > 0), \tag{2.14}$$

which is easily computed via the Levinson algorithm [27]. This so called *maximum entropy solution* is an all-pole or AR solution, and the corresponding modeling filter

$$w(z) = \frac{z^n}{a(z)} \tag{2.15}$$

has all its zeros at the origin.

However, in many applications a wider variety in the choice of zeros are required in the spectral density $\Phi(z)$. To illustrate this point, consider in Figure 2.1 a spectral density in the form of a periodogram determined from a speech signal sampled over 20 ms (in which time interval it represents a stationary process) together with a maximum entropy solution corresponding to $n = 6$. As can be seen the latter yields a rather flat spectrum which is unable to approximate the valleys or the "notches" in the speech spectrum, and therefore in speech synthesis the maximum entropy solution results in artificial speech which sound quite flat. This is a manifestation of the fact
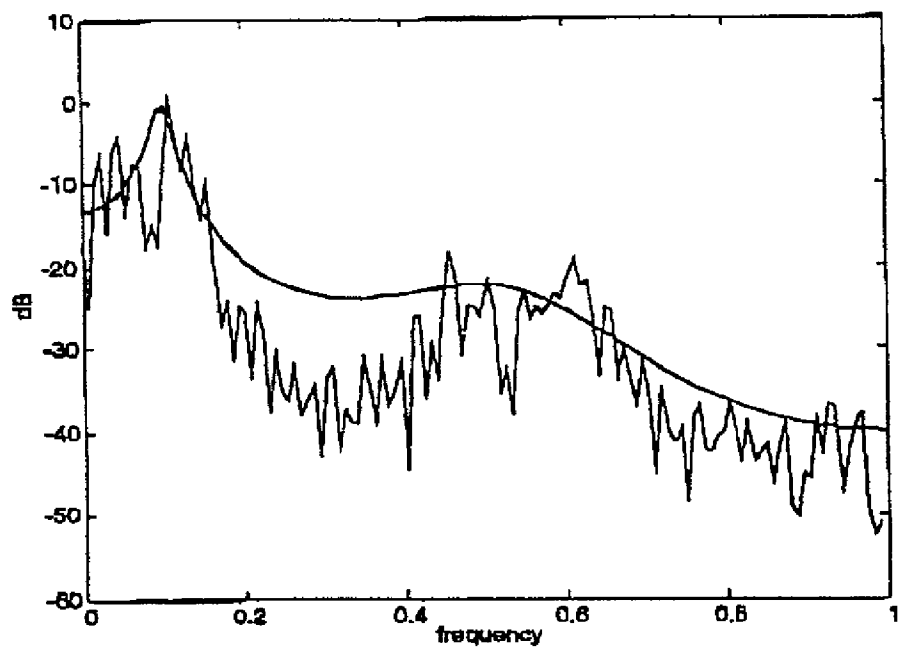
Figure 2.1: Spectral envelope of a maximum entropy solution.

6   C. I. BYRNES, S. V. GUSEV, AND A. LINDQUIST

a corollary of a more general theorem on complementary foliations of the space of all
rational positive real functions of degree at most n.

Theorem 2.1 ([7]). *Given any partial covariance sequence (2.4) and Schur poly-
nomial (2.17), there exists a unique Schur polynomial (2.14) such that (2.16) is a
minimum-phase spectral factor of a spectral density $\Phi(z)$ satisfying*

$$\Phi(z) = c_0 + \sum_{k=1}^{\infty} \hat{c}_k(z^k + z^{-k}),$$

where

$$\hat{c}_k = c_k \quad for \quad i = 1, 2, \ldots, n.$$

*In particular, the solutions of the rational positive extension problem are in one-one
correspondence with self-conjugate sets of n points (counted with multiplicity) lying in
the open unit disc, i.e. with all possible zero structures of modeling filters. Moreover,
this correspondence is bianalytic.*

Consequently, we not only proved Georgiou's conjecture that the family of all ra-
tional covariance extensions of (2.4) of degree at most n is completely parameterized
in terms of the zeros of the corresponding modeling filters $w(z)$, but also that the
modeling filter $w(z)$ depends analytically on the covariance data and the choice of
zeros, a strong form of well-posedness increasing the likelihood of finding a numerical
algorithm.

In fact, both Georgiou's existence proof and our proof of Theorem 2.1 are noncon-
structive. However, in this paper we present for the first time an algorithm which,
given the partial covariance sequence (2.4) and the desired zero polynomial (2.17),
computes the unique pole-polynomial (2.14). This is done via the convex optimization
problem to minimize the value of the function $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}$, defined by

$$\varphi(q_0, q_1, \ldots, q_n) = c_0 q_0 + c_1 q_1 + \cdots + c_n q_n$$
$$- \frac{1}{2\pi} \int_{-\pi}^{\pi} \log Q(e^{i\theta})|\sigma(e^{i\theta})|^2 d\theta, \tag{2.18}$$

over all $q_0, q_1, \ldots, q_n$ such that

$$Q(e^{i\theta}) = q_0 + q_1 \cos\theta + q_2 \cos 2\theta + \cdots + q_n \cos n\theta > 0 \quad \text{for all } \theta. \tag{2.19}$$

In Sections 4 and 5 we show this problem has a unique minimum. In this way we
shall also provide a new and constructive proof of the weaker form of Theorem 2.1
conjectured by Georgiou.

Using this convex optimization problem, a sixth degree modeling filter with zeros
at the appropriate frequencies can be constructed for the speech segment represented
by the periodogram of Figure 2.1. In fact, Figure 2.2 illustrates the same periodogram
together with the spectral density of such a filter. As can be seen this filter yields a
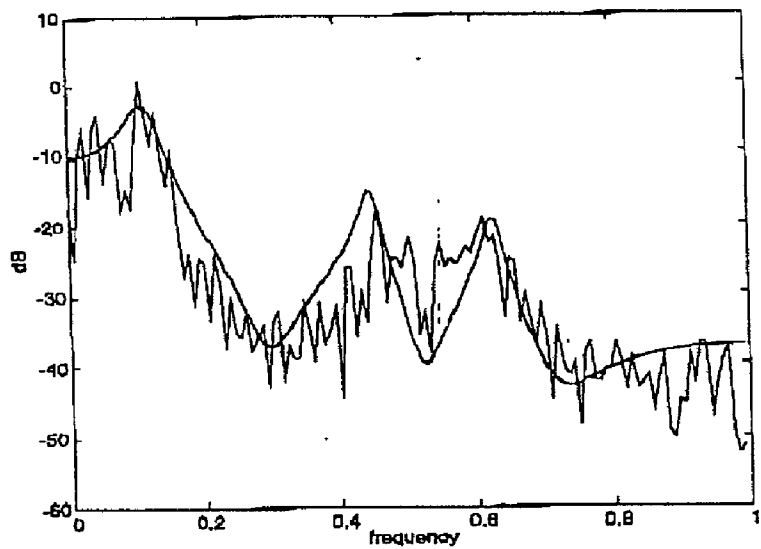much better description of the notches than does the maximum entropy filter.

Figure 2.2: Spectral envelope obtained with appropriate choice of zeros.

We begin by setting up the appropriate spaces. Recall from classical realization theory that a rational function

$$v(z) = \frac{1}{2}\hat{c}_0 + \hat{c}_1 z^{-1} + \hat{c}_2 z^{-2} + \dots$$

of degree n has a representation

$$\hat{c}_k = h' F^{k-1} g \quad k = 1, 2, 3, \dots$$

for some choice of $(F, g, h) \in \mathbb{R}^{n \times n} \times \mathbb{R}^n \times \mathbb{R}^n$. Therefore, if in addition $v(z)$ is strictly positive real, implying that all eigenvalues of F are less than one in modulus, $\hat{c}_k$ tends exponentially to zero as $k \to \infty$. Hence, in particular,

$$\bar{c} := (\hat{c}_0, \hat{c}_1, \hat{c}_2, \dots)$$

must belong to ℓ_1. Moreover, the requirement that (3.1) be a coercive spectral density adds another constraint namely that $\bar{c}$ belongs to the set

$$\mathcal{F} := \{\hat{c} \in \ell_1 \mid \hat{c}_0 + \sum_{k=1}^{\infty} \bar{c}_k (e^{ik\theta} + e^{-ik\theta}) > 0\}. \tag{3.4}$$

Now, let

$$\psi(\bar{c}) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left[\hat{c}_0 + \sum_{k=1}^{\infty} \hat{c}_k (e^{ik\theta} + e^{-ik\theta}) \right] d\theta, \tag{3.5}$$

be a functional $\mathcal{F} \to \mathbb{R}$, and consider the infinite-dimensional convex constrained optimization problem to minimize $\psi(\bar{c})$ over $\mathcal{F}$ given the finite number of constraints (3.2). Thus we have relaxed the optimization problem to allow also for nonrational spectral densities.

Since the optimization problem is convex, the Lagrange function

$$L(\hat{c}, \lambda) = \psi(\bar{c}) + \sum_{k=0}^{n} \lambda_k (\hat{c}_k - c_k) \tag{3.6}$$

has a saddle point [26, p. 458] provided the stationary point lie in the interior of $\mathcal{F}$, and, in this case, the optimal Lagrange vector $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_n) \in \mathbb{R}^{n+1}$ can be determined by solving the *dual problem* to maximize

$$\rho(\lambda) = \min_{\hat{c} \in \mathcal{F}} L(\bar{c}, \lambda). \tag{3.7}$$

To this end, first note that

$$\frac{\partial L}{\partial \hat{c}_k} = -\frac{1}{2\pi} \int_{-\pi}^{\pi} (e^{ik\theta} + e^{-ik\theta}) \Phi^{-1}(e^{i\theta}) d\theta + \lambda_k \quad \text{for } k = 0, 1, 2, \dots, n, \tag{3.8}$$

and that

$$\frac{\partial L}{\partial \hat{c}_k} = -\frac{1}{2\pi} \int_{-\pi}^{\pi} (e^{ik\theta} + e^{-ik\theta}) \Phi^{-1}(e^{i\theta}) d\theta \quad \text{for } k = n+1, n+2, \dots. \tag{3.9}$$

Then, setting the gradient equal to zero, we obtain from (3.9)

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} (e^{ik\theta} + e^{-ik\theta}) \Phi^{-1}(e^{i\theta}) d\theta = 0 \quad \text{for } |k| > n,$$

from which it follows that Φ^{-1} must be a pseudo polynomial

$$Q(z) = q_0 + \tfrac{1}{2}q_1(z + z^{-1}) + \cdots + \tfrac{1}{2}q_n(z^n + z^{-n}) \tag{3.10}$$

of degree at most n, i.e.,

$$\Phi^{-1}(z) = Q(z), \tag{3.11}$$

yielding a spectral density Φ which is rational of at most degree $2n$ and thus belonging to the original (nonrelaxed) class of spectral densities. Likewise we obtain from (3.8)

$$\lambda_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} (e^{ik\theta} + e^{-ik\theta})\Phi^{-1}(e^{i\theta})d\theta \tag{3.12}$$

for $k = 0, 1, 2, \ldots, n$, which together with (3.11) yields

$$\lambda_k = q_k \quad \text{for } k = 0, 1, 2, \ldots, n. \tag{3.13}$$

But, the minimizing $\hat{c}$ is given by

$$\hat{c}_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2}(e^{ik\theta} + e^{-ik\theta})Q(e^{i\theta})^{-1}d\theta \tag{3.14}$$

and consequently

$$\sum_{k=0}^{n} q_k \hat{c}_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} Q(e^{i\theta})Q(e^{i\theta})^{-1}d\theta = 1. \tag{3.15}$$

To determine the optimal (saddle point) Lagrange multipliers we turn to the dual problem. In view of (3.11), (3.13) and (3.15), the dual function is

$$\rho(q) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log Q(e^{i\theta})d\theta + 1 - c'q,$$

where $c \in \mathbb{R}^{n+1}$ is the vector with components $c_0, c_1, \ldots, c_n$. Consequently, the dual problem is equivalent to minimizing

$$\varphi(q) = c'q - \frac{1}{2\pi} \int_{-\pi}^{\pi} \log Q(e^{i\theta})d\theta \tag{3.16}$$

over all $q \in \mathbb{R}^{n+1}$ such that the pseudo polynomial (3.10) is nonnegative on the unit circle, i.e.,

$$Q(e^{i\theta}) > 0 \quad \text{for all } \theta, \tag{3.17}$$

and, if the dual problem has an optimal solution satisfying (3.17), the optimal Q solves the primal problem when inserted into (3.11).

The dual problem to minimize (3.16) given (3.17) is a finite-dimensional convex optimization problem, which is simpler than the original (primal) problem. Clearly it is a special case of the optimization problem (2.18)–(2.19), obtained by setting $|\sigma(e^{i\theta})|^2 = 1$ as required for the maximum entropy solution. Figure 3.1 depicts a typical cost function φ in the case $n = 1$. As seen it is convex and attains its optimum in an interior point so that the spectral density Φ has all its poles in the open unit disc as required. That this is the case in general will be proven in Section 5.

We stress again that the purpose of this section is not primarily to derive an algorithm for the maximum entropy solution, for which we already have the simple
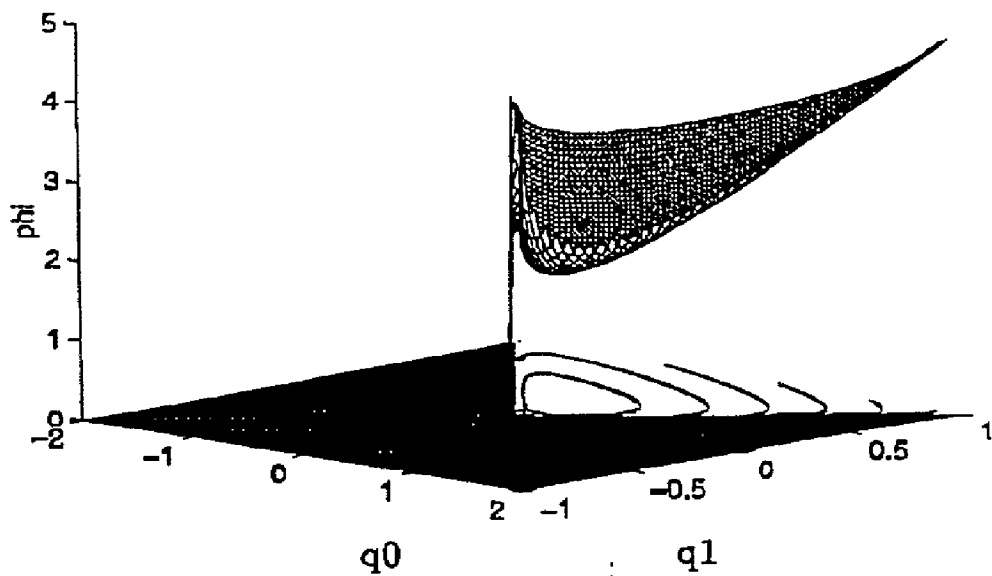
Figure 3.1: A typical cost function $\varphi(q)$ in the case $n = 1$.

CONVEX OPTIMIZATION FOR RATIONAL COVARIANCE EXTENSIONS   11

so the problem is reduced to determining the variables

$$q = \begin{bmatrix} q_0 \\ q_1 \\ \vdots \\ q_n \end{bmatrix} \in \mathbb{R}^{n+1} \qquad (4.5)$$

in the pseudo-polynomial

$$Q(z) = q_0 + \tfrac{1}{2}q_1(z + z^{-1}) + \tfrac{1}{2}q_2(z^2 + z^{-2}) + \cdots + \tfrac{1}{2}q_n(z^n + z^{-n}) \qquad (4.6)$$

so that the conditions (4.3) and

$$Q(e^{i\theta}) > 0 \quad \text{for all } \theta \in [-\pi, \pi] \qquad (4.7)$$

are satisfied.

Now, consider the convex functional $\varphi(q) : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by

$$\varphi(q) = c'q - \frac{1}{2\pi} \int_{-\pi}^{\pi} \log Q(e^{i\theta}) |\sigma(e^{i\theta})|^2 d\theta. \qquad (4.8)$$

Our motivation in defining $\varphi(q)$ comes in part from the desire to introduce a barrier-like term, as is done in interior point methods, and in part from our analysis of the maximum entropy method in the previous section. As it turns out, by a theorem of Szegö the logarithmic integrand is in fact integrable for nonzero Q having zeros on the boundary of the unit circle, so that $\varphi(q)$ does not become infinite on the boundary of the convex set. On the other hand, $\varphi(q)$ is a natural generalization of the functional (3.16) in Section 3, since it specializes to (3.16) when $|\sigma(e^{i\theta})|^2 \equiv 1$ as for the maximum entropy solution. As we shall see, minimizing (4.8) yields precisely via (4.4) the unique $a(z)$ which corresponds to $\sigma(z)$.

It is clear that if $q \in \mathcal{D}_n^+$ where

$$\mathcal{D}_n^+ = \{q \in \mathbb{R}^{n+1} \mid Q(z) > 0 \quad \text{for } |z| = 1\}, \qquad (4.9)$$

then $\varphi(q)$ is finite. Moreover, $\varphi(q)$ is also finite when $Q(z)$ has finitely many zeros on the unit circle, as can be seen from the following lemma.

Lemma 4.1. *The functional $\varphi(q)$ is finite and continuous at any $q \in \overline{\mathcal{D}_n^+}$ except at zero. The functional is infinite, but continuous, at $q = 0$. Moreover, φ is a C^∞ function on $\mathcal{D}_n^+$.*

Proof. We want to prove that $\varphi(q)$ is finite when $q \neq 0$. Then the rest follows by inspection. Clearly, $\varphi(q)$ cannot take the value $-\infty$; hence it remains to prove that $\varphi(q) < \infty$. Since $q \neq 0$,

$$\mu := \max_\theta Q(e^{i\theta}) > 0.$$

Then, setting $P(z) := \mu^{-1}Q(z)$,

$$\log P(e^{i\theta}) \leq 0 \qquad (4.10)$$

and

$$\varphi(q) = c'q - \frac{1}{2\pi} \log \mu \int_{-\pi}^{\pi} |\sigma(e^{i\theta})|^2 d\theta - \frac{1}{2\pi} \int_{-\pi}^{\pi} \log P(e^{i\theta}) |\sigma(e^{i\theta})|^2 d\theta,$$

and hence the question of whether $\varphi(q) < \infty$ is reduced to determining whether

$$-\int_{-\pi}^{\pi} \log P(e^{i\theta})|\sigma(e^{i\theta})|^2 d\theta < \infty.$$

But, since $|\sigma(e^{i\theta})|^2 \le M$ for some bound M, this follows from

$$\int_{-\pi}^{\pi} \log P(e^{i\theta})d\theta > -\infty, \tag{4.11}$$

which is the well-known Szegö condition: (4.11) is a necessary and sufficient condition for $P(e^{i\theta})$ to have a stable spectral factor [17]. But, since $P(z)$ is a symmetric pseudo-polynomial which is nonnegative on the unit circle, there is a polynomial $\pi(z)$ such that $\pi(z)\pi(z^{-1}) = P(z)$. But then $w(z) = \frac{\pi(z)}{z^n}$ is a stable spectral factor, and hence (4.11) holds. $\square$

Lemma 4.2. *The functional $\varphi(q)$ is strictly convex and defined on a closed, convex domain.*

Proof. We first note that $q = 0$ is an extreme point, but it can never be a minimum of φ since $\varphi(0)$ is infinite. In particular, in order to check the strict inequality

$$\varphi(\lambda q^{(1)} + (1-\lambda)q^{(2)}) < \lambda\varphi(q^{(1)}) + (1-\lambda)\varphi(q^{(2)}), \tag{4.12}$$

where one of the arguments is zero, one need only consider the case that one of $q^{(1)}$ or $q^{(2)}$ is zero, in which case the strict inequality holds. We can now assume that none of the arguments is zero, in which case the strict inequality in (4.12) follows from the strict concavity of the logarithm. Finally, it is clear that $\overline{\mathcal{D}_n^+}$ is a closed convex subset. $\square$

Lemma 4.3. *Let $q \in \overline{\mathcal{D}_n^+}$, and suppose $q \ne 0$. Then $c'q > 0$.*

Proof. Consider an arbitrary covariance extension of c such as, for example, the maximum entropy extension, and let $\Phi(z)$ be the corresponding spectral density (2.9). Then c is given by (2.3), which may also be written

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2}(e^{ik\theta} + e^{-ik\theta})\Phi(e^{i\theta})d\theta; \quad k = 0, 1, \ldots, n.$$

Therefore, in view of (4.6),

$$c'q = \frac{1}{2\pi}\int_{-\pi}^{\pi} Q(e^{ik\theta})\Phi(e^{i\theta})d\theta, \tag{4.13}$$

which is positive whenever $Q(z) \ge 0$ on the unit circle and $q \ne 0$. $\square$

Proposition 4.4. *For all $r \in \mathbb{R}$, $\varphi^{-1}(-\infty, r]$ is compact. Thus φ is proper (i.e., $\varphi^{-1}(K)$ is compact whenever K is compact) and bounded from below.*

Proof. Suppose $q^{(k)}$ is a sequence in $M_r := \varphi^{-1}(-\infty, r]$. It suffices to show that $q^{(k)}$ has a convergent subsequence. Each $Q^{(k)}$ may be factored as

$$Q^{(k)}(z) = \lambda_k \bar{a}^{(k)}(z)\bar{a}^{(k)}(z^{-1}) = \lambda_k \bar{Q}^{(k)}(z),$$

where λ_k is positive and $\bar{a}^{(k)}(z)$ is a monic polynomial all of whose roots lie in the closed unit disc. The corresponding sequence of the (unordered) set of n roots of

each $\bar{a}^{(k)}(z)$ has a convergent subsequence, since all (unordered) sets of roots lie in the closed unit disc. Denote by $\bar{a}(z)$ the monic polynomial of degree n which vanishes at this limit set of roots. By reordering the sequence if necessary, we may assume the sequence $a^{(k)}(z)$ tends to $\bar{a}(z)$. Therefore the sequence $q^{(k)}$ has a convergent subsequence if and only if the sequence λ_k does, which will be the case provided the sequence λ_k is bounded from above and from below away from zero. Before proving this, we note that the sequences $c'\bar{q}^{(k)}$, where $\bar{q}^{(k)}$ is the vector corresponding to the pseudo-polynomial $\bar{Q}^{(k)}$, and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\bar{Q}^{(k)}(e^{i\theta})|\sigma(e^{i\theta})|^2 d\theta \qquad (4.14)$$

are both bounded from above and from below respectively away from zero and $-\infty$. The upper bounds come from the fact that $\{\bar{a}^{(k)}(z)\}$ are Schur polynomials and hence have their coefficients in the bounded Schur region. As for the lower bound of $c'\bar{q}^{(k)}$, note that $c'\bar{q}^{(k)} > 0$ for all k (Lemma 4.3) and $c'\bar{q}^{(k)} \to \alpha > 0$. In fact, $\bar{Q}^{(k)}(e^{i\theta}) \to |\bar{a}(e^{i\theta})|^2$, where $\bar{a}(z)$ has all its zeros in the closed unit disc, and hence it follows from (4.13) that $\alpha > 0$. Then, since $\varphi(q) < \infty$ for all $q \in \overline{\mathcal{D}_n^+}$ except $q = 0$ (Lemma 4.1), (4.14) is bounded away from $-\infty$. Next, observe that

$$\varphi(q^{(k)}) = \lambda_k c'\bar{q}^{(k)} - \frac{1}{2\pi}\log\lambda_k\int_{-\pi}^{\pi}|\sigma(e^{i\theta})|^2 d\theta - \frac{1}{2\pi}\int_{-\pi}^{\pi}\log\bar{Q}^{(k)}(e^{i\theta})|\sigma(e^{i\theta})|^2 d\theta.$$

From this we can see that if a subsequence of λ_k were to tend to zero, then $\varphi(q^{(k)})$ would exceed r. Likewise, if a subsequence of λ_k were to tend to infinity, φ would exceed r, since linear growth dominates logarithmic growth. $\square$

5. Interior critical points and solutions of the rational covariance extension problem

In the previous section, we showed that φ has compact sublevel sets in $\overline{\mathcal{D}_n^+}$, so that φ achieves a minimum. Moreover, since φ is strictly convex and $\overline{\mathcal{D}_n^+}$ is convex, such a minimum is unique. We record these observations in following statement.

Proposition 5.1. *For each partial covariance sequence c and each Schur polynomial $\sigma(z)$, the functional φ has a unique minimum on $\overline{\mathcal{D}_n^+}$.*

In this paper we consider a question which is of independent interest, the question of whether φ achieves its minimum at an interior point. The next result describes an interesting systems-theoretic consequence of the existence of such interior minima.

Theorem 5.2. *Fix a partial covariance sequence c and a Schur polynomial $\sigma(z)$. If $\bar{q} \in \mathcal{D}_n^+$ is a minimum for φ, then*

$$\hat{Q}(z) = a(z)\hat{a}(z^{-1}), \qquad (5.1)$$

where $a(z)$ is the solution of the rational covariance extension problem.

Proof. Suppose that $\bar{q} \in \mathcal{D}_n^+$ is a minimum for φ. Then

$$\frac{\partial\varphi}{\partial q_k}(\bar{q}) = 0 \quad \text{for } k = 0, 1, 2, \ldots, n, \qquad (5.2)$$

Differentiating inside the integral, which is allowed due to uniform convergence, (5.2) yields

$$c_k - \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2}(e^{ik\theta} + e^{-ik\theta})\frac{|\sigma(e^{i\theta})|^2}{\hat{Q}(e^{i\theta})}d\theta = 0 \quad \text{for } k = 0, 1, \ldots, n,$$

where $\hat{Q}(z)$ is the pseudo-polynomial (4.6) corresponding to $\hat{q}$, or, equivalently,

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi}e^{ik\theta}\frac{|\sigma(e^{i\theta})|^2}{\hat{Q}(e^{i\theta})}d\theta \quad \text{for } k = 0, 1, \ldots, n, \tag{5.3}$$

which is precisely the interpolation condition (4.3)–(4.4), provided (5.1) holds. $\square$

As a corollary of this theorem we have that the gradient of φ at *any* $\bar{q} \in \mathcal{D}_n^+$ is given by

$$\frac{\partial\varphi}{\partial q_k}(\bar{q}) = c_k - \tilde{c}_k \tag{5.4}$$

where

$$\tilde{c}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi}e^{ik\theta}\frac{|\sigma(e^{i\theta})|^2}{\bar{Q}(e^{i\theta})}d\theta \quad k = 0, 1, 2, \ldots, n \tag{5.5}$$

is the partial covariance sequence corresponding to a process with spectral density

$$\tilde{\Phi}(e^{i\theta}) = \frac{|\sigma(e^{i\theta})|^2}{\bar{Q}(e^{i\theta})},$$

where $\bar{Q}(z)$ is the pseudo-polynomial corresponding to $\bar{q}$. The gradient is thus the difference between the true and calculated partial covariance sequence.

We now state the converse result, underscoring our interest in this particular convex optimization problem.

Theorem 5.3. *For each partial covariance sequence c and each Schur polynomial* $\sigma(z)$, *suppose that* $a(z)$ *gives a solution to the rational covariance extension problem. If*

$$\hat{Q}(z) = a(z)a(z^{-1}), \tag{5.6}$$

then the corresponding $(n+1)$*-vector* $\hat{q}$ *lies in* $\mathcal{D}_n^+$ *and is a unique minimum for* φ.

Proof. Let $a(z)$ be the solution of the rational covariance extension problem corresponding to c and $\sigma(z)$, and let $\hat{Q}(z)$ be given by (5.6). Then c satisfies the interpolation condition (5.3), which is equivalent to (5.2), as seen from the proof of Theorem 5.2. But, since $a(z)$ is a Schur polynomial, $\hat{Q}(z) > 0$ on the unit circle, and thus $\hat{q} \in \mathcal{D}_n^+$. Since φ is strictly convex on $\mathcal{D}_n^+$, (5.3) implies that $\hat{q}$ is a unique minimum for φ. $\square$

Since the existence of a solution to the rational covariance extension problem has been established in [14] (see also [7]), one does in fact know the existence of interior minima for this convex optimization problem. On the other hand, we know from Proposition 5.1 that φ has a minimum for some $\bar{q} \in \overline{\mathcal{D}_n^+}$, so to show that φ has a minimum in the interior $\mathcal{D}_n^+$ it remains to prove the following lemma.

Lemma 5.4. *The functional* φ *never attains a minimum on the boundary* $\partial\mathcal{D}_n^+$.

Proof. Denoting by $D_p\varphi(q)$ the directional derivative of φ at q in the direction p, it is easy to see that

$$D_p\varphi(q) := \lim_{\epsilon \to 0} \frac{\varphi(q + \epsilon p) - \varphi(q)}{\epsilon} \tag{5.7}$$

$$= c'p - \int_{-\pi}^{\pi} \frac{P(e^{i\theta})}{Q(e^{i\theta})} |\sigma(e^{i\theta})|^2 d\theta, \tag{5.8}$$

where $P(z)$ is the pseudo-polynomial

$$P(z) = p_0 + \tfrac{1}{2}p_1(z + z^{-1}) + \tfrac{1}{2}p_2(z^2 + z^{-2}) + \cdots + \tfrac{1}{2}p_n(z^n + z^{-n})$$

corresponding to the vector $p \in \mathbb{R}^{n+1}$. In fact,

$$\frac{\log(Q + \epsilon P) - \log Q}{\epsilon} = \frac{P}{Q} \log\left[(1 + \epsilon\frac{P}{Q})^{\frac{1}{\epsilon}\frac{Q}{P}}\right] \to \frac{P}{Q}$$

as $\epsilon \to +0$, and hence (5.7) follows by dominated convergence.

Now, let $q \in \mathcal{D}_n^+$ and $\bar{q} \in \partial\mathcal{D}_n^+$ be arbitrary. Then the corresponding pseudo-polynomials Q and $\bar{Q}$ have the properties

$$Q(e^{i\theta}) > 0 \quad \text{for all } \theta \in [-\pi, \pi]$$

and

$$\bar{Q}(e^{i\theta}) \geq 0 \quad \text{for all } \theta \text{ and } \bar{Q}(e^{i\theta_0}) = 0 \text{ for some } \theta_0.$$

Since $q_\lambda := \bar{q} + \lambda(q - \bar{q}) \in \mathcal{D}_n^+$ for $\lambda \in (0, 1]$, we also have for $\lambda \in (0, 1]$ that

$$Q_\lambda(e^{i\theta}) := \bar{Q}(e^{i\theta}) + \lambda[Q(e^{i\theta}) - \bar{Q}(e^{i\theta})] > 0, \quad \text{for all } \theta \in [-\pi, \pi],$$

and we may form the directional derivative

$$D_{\bar{q}-q}\varphi(q_\lambda) = c'(\bar{q} - q) + \int_{-\pi}^{\pi} h_\lambda(\theta) d\theta, \tag{5.9}$$

where

$$h_\lambda(\theta) = \frac{Q(e^{i\theta}) - \bar{Q}(e^{i\theta})}{Q_\lambda(e^{i\theta})} |\sigma(e^{i\theta})|^2.$$

Now,

$$\frac{d}{d\lambda} h_\lambda(\theta) = \frac{[Q(e^{i\theta}) - \bar{Q}(e^{i\theta})]^2}{Q_\lambda(e^{i\theta})^2} |\sigma(e^{i\theta})|^2 \geq 0,$$

and hence $h_\lambda(\theta)$ is a monotonely nondecreasing function of λ for all $\theta \in [-\pi, \pi]$. Consequently h_λ tends pointwise to h_0 as $\lambda \to 0$. Therefore,

$$\int_{-\pi}^{\pi} h_\lambda(\theta) d\theta \to +\infty \quad \text{as } \lambda \to 0. \tag{5.10}$$

In fact, if

$$\int_{-\pi}^{\pi} h_\lambda(\theta) d\theta \to \alpha < \infty \quad \text{as } \lambda \to 0, \tag{5.11}$$

then $\{h_\lambda\}$ is a Cauchy sequence in $L^1(-\pi, \pi)$ and hence has a limit in $L^1(-\pi, \pi)$ which must equal h_0 a.e. But h_0, having poles in $[-\pi, \pi]$, is not summable and hence, as claimed, (5.11) cannot hold.

Consequently, by virtue of (5.9),

$$D_{q-\bar{q}}\varphi(q_\lambda) \to +\infty \quad \text{as } \lambda \to 0$$

for all $q \in \mathcal{D}_n^+$ and $\bar{q} \in \partial\mathcal{D}_n^+$, and hence, in view of Lemma 26.2 in [28], φ is essentially smooth. Then it follows from Theorem 26.3 in [28] that the subdifferential of φ is empty on the boundary of $\mathcal{D}_n^+$, and therefore φ cannot have a minimum there. $\square$

Thus we have proven the following result.

Theorem 5.5. *For each partial covariance sequence c and each Schur polynomial $\sigma(z)$, there exists an $(n+1)$-vector $\bar{q}$ in $\mathcal{D}_n^+$ which is a minimum for φ.*

Consequently, by virtue of Theorem 5.2, there does exist a solution to the rational covariance extension problem for each partial covariance sequence and zero polynomial $\sigma(z)$, and, in view of Theorem 5.3, this solution is unique.

These theorems have the following corollary.

Corollary 5.6 (Georgiou's Conjecture) *For each partial covariance sequence c and each Schur polynomial $\sigma(z)$, there is a unique Schur polynomial $a(z)$ such that (4.1) and (4.2) hold.*

Hence, we have given an independent proof of the weaker version of Theorem 2.1 conjectured by Georgiou, but not of the stronger version of [7] which states that the problem is well-posed in the sense that the one-one correspondence between $\sigma(z)$ and $a(z)$ is a diffeomorphism.

6. Some numerical examples

Given an arbitrary partial covariance sequence $c_0, c_1, \ldots, c_n$ and an arbitrary zero polynomial $\sigma(z)$, the constructive proof of Georgiou's conjecture provides algorithmic procedures for computing the corresponding unique modeling filter, which are based on the convex optimization problem to minimize the functional (2.18) over all $q_0, q_1, \ldots, q_n$ such that (2.19) holds.

In general such procedures will be based on the gradient of the cost functional φ, which, as we saw in Section 5, is given by

$$\frac{\partial\varphi}{\partial q_k}(q_0, q_1, \ldots, q_n) = c_k - \bar{c}_k \tag{6.1}$$

where

$$\bar{c}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\frac{|\sigma(e^{i\theta})|^2}{Q(e^{i\theta})}d\theta \quad \text{for } k = 0, 1, 2, \ldots, n \tag{6.2}$$

are the covariances corresponding to a process with spectral density

$$\frac{|\sigma(e^{i\theta})|^2}{Q(e^{i\theta})} = \bar{c}_0 + 2\sum_{k=1}^{\infty}\bar{c}_k\cos(k\theta). \tag{6.3}$$

The gradient is thus the difference between the given partial covariance sequence $c_0, c_1, \ldots, c_n$ and the partial covariance sequence corresponding to the choice of variables $q_0, q_1, \ldots, q_n$ at which the gradient is calculated. The minimum is attained when this difference is zero.

Figure 6.1

44

Figure 6.2

CONVEX OPTIMIZATION FOR RATIONAL COVARIANCE EXTENSIONS   19
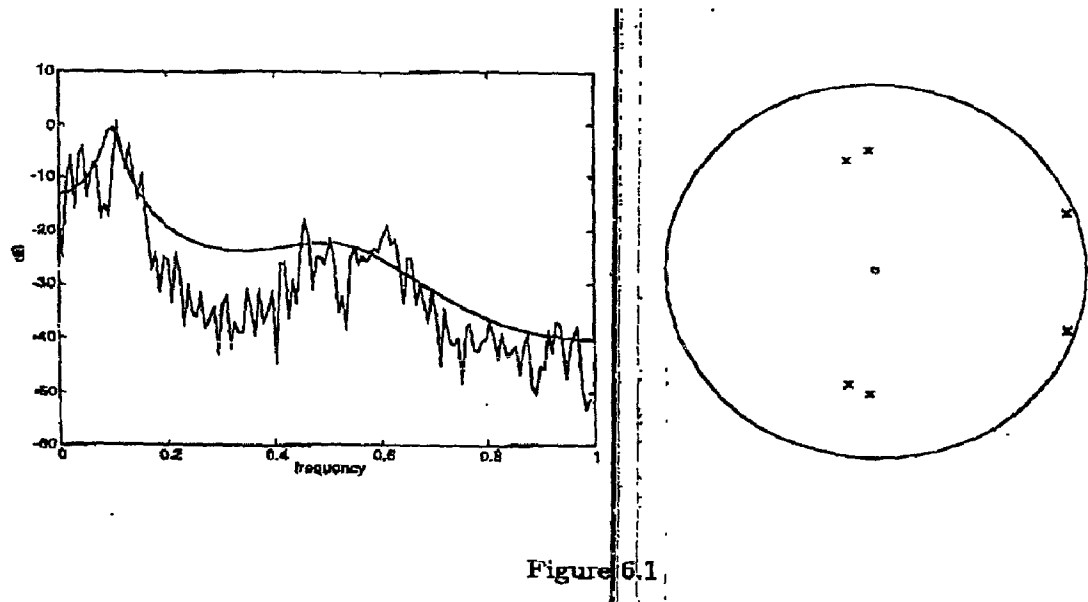
References

1. N. I. Akhiezer, *The Classical Moment Problem*, Hafner, 1965.

2. M. Aoki, *State Space Modeling of Time Series*, Springer-Verlag, 1987.

3. C. G. Bell, H. Fujisaki, J. M. Heinz, K. N. Stevens and A. S. House, *Reduction of Speech Spectra by Analysis-by-Synthesis Techniques*, J. Acoust. Soc. Am. **33** (1961), 1725–1736.

4. C. I. Byrnes and A. Lindquist, *On the geometry of the Kimura-Georgiou parameterization of modelling filter*, Inter. J. of Control **50** (1989), 2301–2312.

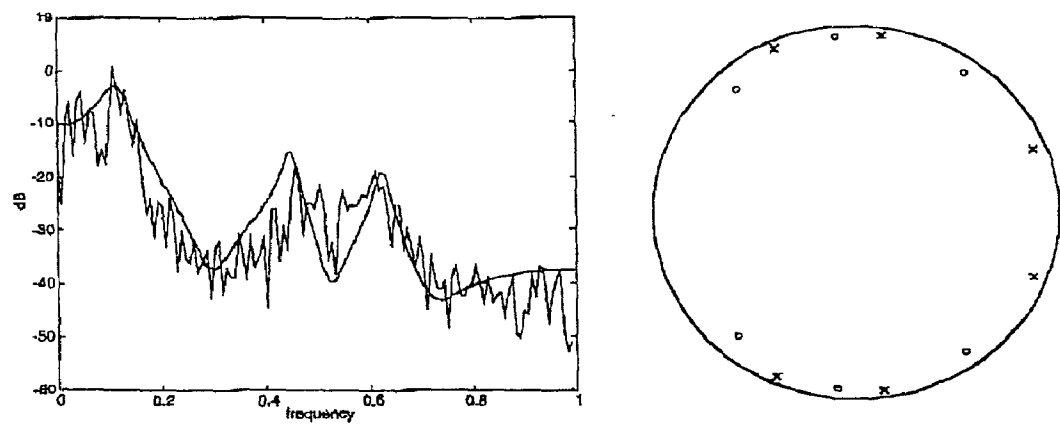5. C. I. Byrnes and A. Lindquist, *Toward a solution of the minimal partial stochastic realization problem*, Comptes Rendus Acad. Sci. Paris, t. 319, Série I (1994), 1231–1236.

6. C. I. Byrnes and A. Lindquist, *On the partial stochastic realization problem*, IEEE Trans. Automatic Control AC-42 (1997), 1049–1070.

7. C. I. Byrnes, A. Lindquist, S. V. Gusev, and A. S. Mateev, *A complete parametrization of all positive rational extensions of a covariance sequence*, IEEE Trans. Automatic Control AC-40 (1995), 1841–1857.

8. J. A. Cadzow, *Spectral estimation: An overdetermined rational model equation approach*, Proceedings IEEE **70** (1982), 907–939.

9. C. Carathéodory, *Über den Variabilitätsbereich der Koeffizienten von Potenzreihen, die gegebene Werte nicht annehmen*, Math. Ann. **64** (1907), 95–115.

10. C. Carathéodory, *Über den Variabilitätsbereich der Fourier'schen Konstanten von positiven harmonischen Functionen*, Rend. di Palermo **32** (1911), 193–217.

11. Ph. Delsarte, Y. Genin, Y. Kamp and P. van Dooren, *Speech modelling and the trigonometric moment problem*, Philips J. Res. **37** (1982), 277–292.

12. P. Enqvist, forthcoming PhD disertation, Division of Optimization and Systems Theory, Royal Institute of Technology, Stockholm, Sweden.

13. T. T. Georgiou, *Partial realization of covariance sequences*, CMST, Univ. Florida, Gainesville, 1983.

14. T. T. Georgiou, *Realization of power spectra from partial covariance sequences*, IEEE Transactions Acoustics, Speech and Signal Processing ASSP-35 (1987), 438–449.

15. Ya. L. Geronimus, *Orthogonal polynomials*, Consultants Bureau, New York, 1961.

16. U. Grenander and G. Szegö, *Toeplitz forms and their applications*, Univ. California Press, 1958.

17. U. Grenander and M. Rosenblatt, *Statistical Analysis of Stationary Time Series*, Almqvist & Wiksell, Stockholm, 1956.

18. R. E. Kalman, *Realization of covariance sequences*, Proc. Toeplitz Memorial Conference (1981), Tel Aviv, Israel, 1981.

19. S. A. Kassam and H. V. Poor, *Robust techniques for signal processing*, Proceedings IEEE **73** (1985), 433–481.

20. S. M. Kay and S. L. Marple, Jr., *Spectrum Analysis–A modern perspective*, Proceedings IEEE **69** (1981), 1380–1419.

21. H. Kimura, *Positive partial realization of covariance sequences*, Modelling, Identification and Robust Control (C. I. Byrnes and A. Lindquist eds.), North-Holland, 1987, pp. 499–513.

22. A. Lindquist and G. Picci, *Canonical correlation analysis, approximate covariance extension, and identification of stationary time series*, Automatica **32** (May 1996), 709–733.

23. D. G. Luenberger, *Linear and Nonlinear Programming* (Second Edition), Addison-Wesley Publishing Company, Reading, Mass., 1984.

24. J. D. Markel and A. H. Gray, *Linear Prediction of Speech*, Springer Verlag, Berlin, 1976.

25. J. Makhoul, *Linear prediction: A tutorial review*, Proceedings IEEE **63** (1975), 561–580.

26. M. Minoux, Jr., *Mathematical Programming: Theory and Algorithms*, John Wiley and Sons, 1986.

27. B. Porat, *Digital Processing of Random Signals*, Prentice Hall, 1994.

28. R. T. Rockafellar, *Convex Analysis*, Princeton University Press, 1970.

29. L. R. Rabiner and R. W. Schafer, *Digital Processing of Speech Signals*, Prentice Hall, Englewood Cliffs, N.J., 1978.

30. I. Schur, *On power series which are bounded in the interior of the unit circle I and II*, Journal für die reine und angewandte Mathematik **148** (1918), 122–145.

20   C. I. BYRNES, S. V. GUSEV, AND A. LINDQUIST

31. O. Toeplitz, *Über die Fouriersche Entwicklung positiver Funktionen*, Rendiconti del Circolo Matematico di Palermo **32** (1911), 191–192.

32. P. van Overschee and B. De Moor, *Subspace algorithms for stochastic identification problem*, IEEE Trans. Automatic Control **AC-27** (1982), 382–387.

DEPARTMENT OF SYSTEMS SCIENCE AND MATHEMATICS, WASHINGTON UNIVERSITY, ST. LOUIS, MISSOURI 63130, USA

DEPARTMENT OF MATHEMATICS AND MECHANICS, ST. PETERSBURG UNIVERSITY, ST. PE-TERSBURG 198904, RUSSIA

DIVISION OF OPTIMIZATION AND SYSTEMS THEORY, ROYAL INSTITUTE OF TECHNOLOGY, 100 44 STOCKHOLM, SWEDEN

APPENDIX D

..

A GENERALIZED ENTROPY CRITERION FOR NEVANLINNA-PICK INTERPOLATION: A CONVEX OPTIMIZATION APPROACH TO CERTAIN PROBLEMS IN SYSTEMS AND CONTROL*

CHRISTOPHER I. BYRNES†, TRYPHON T. GEORGIOU§, AND ANDERS LINDQUIST‡

June 29, 1998

ABSTRACT. In this paper we present a generalized entropy critierion for solving the rational Nevanlinna-Pick problem with degree constraint for $n + 1$ interpolating conditions. We also show that the primal problem of maximizing this entropy gain has a very well-behaved dual problem resulting in convex optimization scheme, generalizing that of [10], for finding all solutions of the Nevanlinna-Pick interpolation problem which are positve real, and rational of degree less than or equal to n. This criterion is defined in a form parameterized by an arbitrary choice of a monic Schur polynomial as suggested in [25, 26] and recently verified in [9, 10, 11] for the rational covariance extension problem and [27] for Nevanlinna-Pick interpolation. Our interest in this convex optimization problem is therefore twofold: as a starting point for the computation of explicit solutions to the rational Nevanlinna-Pick problem in terms of a design parameter, and as a means of providing a variational proof of the recent complete parameterization of all solutions of this problem in terms of Schur polynomials. From the optimization problem we design an algorithm which is implemented in state space form and applied to several problems in systems and control, such as sensitivity minimization in $\mathcal{H}^\infty$ control, maximal power transfer, simultaneous stabilization and spectral estimation.

1. Introduction

A general interpolation problem consists of developing conditions for the existence of, as well as a parameterization of, meromorphic solutions to the following interpolation problem: Given $n + 1$ points z_k, for $k = 0, 1, ..., n$ located in a specified region of the complex plane and $n + 1$ desired complex values w_k, find all meromorphic functions f in a given class which satisfy

$$f(z_k) = w_k \quad \text{for } k = 0, 1, ..., n. \tag{1.1}$$

While Lagrange interpolation gives a particular solution to this problem, there has been a remarkable literature developed to solve this problem within special classes

2   C. I. BYRNES, T. T. GEORGIOU, AND A. LINDQUIST

of functions having certain positivity properties. In particular, seminal contributions of Carathéodory, Nevanlinna, Pick, Toeplitz, Schur and others have given solutions within the classes of meromorphic functions which map, for example, the unit disc (or its complement) into the closed left-half plane. These Nevanlinna-Pick interpolation problems have also been reinterpreted, and vastly extended, in an operator theoretic setting; see, e.g., [46, 33, 3, 21]. Indeed, the research developed for interpolation problems of this kind has had substantial impact on classical function theory, harmonic analysis, potential theory, probability, and operator theory.

The same classes of functions, known in the circuit and systems literature as positive real, or bounded real, have long played a fundamental role in describing the impedance of RLC circuits, in formalizing the underlying stability mechanisms relating to the dissipation of energy in circuits and quite general linear and nonlinear systems, and in formulating the positivity of probability measures in stochastic systems theory. For these reasons, problems involving interpolation by positive real functions play an important role in circuit theory [52, 16, 33], robust stabilization and control [48, 49, 54, 39, 37, 28, 18], signal processing [25], speech synthesis [17, 9, 10, 11], and stochastic systems theory [35, 8, 7].

In these contexts, then, the essence of Nevanlinna-Pick theory is directly applicable. However, it is also important that the interpolating function be rational, and this presents some new challenges which need to be incorporated systematically into any useful enhancement of the classical theory. While the Nevanlinna recursion algorithm, and the resulting parameterization of all positive real interpolants in terms of a "free" function, can be used to generate certain rational solutions, it is also important to parameterize all rational solutions of a given degree, for example, n. It will, of course, be crucial in applications to have an effective computational scheme to generate the rational, positive real interpolants of degree at most n. To this end, in this paper we derive a generalized entropy criterion for the problem of rational Nevanlinna-Pick interpolation. As a primal problem, one is led to an optimization problem in infinitely many variables, a problem which has, however, a dual problem which is convex in finitely many variables and for which the interior minimum corresponds precisely to a solution of the Nevanlinna-Pick problem with degree constraints. Moreover, the entropy integral incorporates an arbitrary choice of n points inside the unit disc as "free" parameters, in a natural systems-theoretic fashion as in [25, 26], so that through convex optimization we are able to obtain all solutions of the Nevanlinna-Pick with degree constraints as a function of the zeros inside the unit circle of an associated density function of degree n.

In Section 2 we describe the principal results about the Nevanlinna-Pick problem with degree constraints, and in Section 3 we are setting notation which we shall need throughout. The main results of the paper are then stated in Section 4, in which we define a maximum entropy criterion, generalized to incorporate the data in the rational Nevanlinna-Pick problem. We demonstrate that the infinite-dimensional optimization problem for determining this solution has a simple finite-dimensional dual, which in turn is a generalization of the optimization problem in [10]. The proof of these theorems are given in Section 5 together with an analysis of the dual problem. In fact, the dual problem amounts to minimizing a nonlinear, strictly convex functional, defined on a closed convex set naturally related to the Nevanlinna-Pick problem with degree constraints.

Along similar lines as in [10], we first show that any solution to this problem lies in the interior of this convex set and that, conversely, an interior minimum of this convex functional will correspond to the unique solution of the Nevanlinna-Pick problem. Concerning the existence of a minimum, we show that this functional is proper and bounded below, i.e., that the sublevel sets of this functional are compact. From this, it follows that there exists a minimum. Since uniqueness follows from strict convexity of the functional, the central issue which needs to be addressed in order to solve the rational Nevanlinna-Pick problem turns out to be whether, in fact, this minimum is an interior point. Indeed, the dual problem contains a barrier-like term, as is the case in interior point methods. However, in contrast to interior point methods, the barrier function considered here does not become infinite on the boundary of our closed convex set. Nonetheless, we are able to show that the gradient, rather than the value, of the convex functional in the dual problem becomes infinite on the boundary. The existence of an interior point which minimizes the functional then follows from this observation.

In Section 6 we outline a computational procedure for solving the dual problem, and hence the Nevanlinna-Pick interpolation with degree constraints. In the special case of real interpolants, in Section 7 we develop a state-space procedure, which has the potential to allow extensions to the multivariable case.

Finally, in Section 8, the algorithm is applied to several problems in systems and control, such as sensitivity minimization in $\mathcal{H}^\infty$ control, maximal power transfer, simultaneous stabilization and spectral estimation.

2. The Nevanlinna-Pick interpolation problem with degree constraint

Given two sets of $n+1$ points in $\mathbb{D}^c := \{z \mid |z| > 1\}$ and $\mathbb{C}$ respectively,

$$\mathcal{Z} := \{z_k \mid k = 0, 1, \ldots, n\} \quad \text{and} \quad \mathcal{W} := \{w_k \mid k = 0, 1, \ldots, n\},$$

we seek a parameterization of all functions $f(z)$ satisfying

(i) the interpolation conditions

$$f(z_k) = w_k \quad \text{for } k = 0, 1, \ldots, n, \tag{2.1}$$

(ii) being analytic and having nonnegative real part in $\mathbb{D}^c$, i.e., being positive real, and

(iii) being rational of at most (McMillan) degree n.

Moreover, we require a constructive procedure for computing specific such solutions. This problem will be referred to as the *Nevanlinna-Pick problem with degree constraint*. For future reference, the class of functions satisfying condition (ii), also known as Carathéodory functions, will be denoted by $\mathcal{C}$. Moreover, we denote by $\mathcal{C}_0$ the subclass of *strictly positive real functions*, whose domain of analyticity includes the unit circle and have positive real part.

Requiring only condition (i) amounts to standard Lagrange interpolation, the solution of which is well-known. Adding condition (ii) yields a classical problem in complex analysis, namely the Nevanlinna-Pick interpolation problem. This problem plays a central role in $\mathcal{H}^\infty$ control, simultaneous stabilization, power transfer in circuit theory, model reduction and signal processing. The McMillan degree of

4   C. I. BYRNES, T. T. GEORGIOU, AND A. LINDQUIST

the interpolant relates to the dimension of a corresponding dynamical system, and therefore condition (iii) becomes important.

For the classical Nevanlinna-Pick problem, there exists a solution if and only if an associated *Pick matrix* is positive semidefinite [51, 46]. In case all z_k's are distinct the Pick matrix is given by

$$P = \left[\frac{w_k + \bar{w}_\ell}{1 - \bar{z}_k^{-1} z_\ell^{-1}} \right]_{k,\ell=0}^n.$$

If points in $\mathcal{Z}$ are not distinct, the interpolation conditions involve derivatives of $f(z)$. For example, if $z_k = z_1$ $(k = 2, \ldots, j)$, then these conditions become $\frac{1}{(k-1)!} f^{(k-1)}(z_1) = w_k$, where $f^{(k-1)}(z)$ denotes the $(k-1)$th derivative. In the special case where all the points $z_k, k = 0, 1, \ldots, n$ coincide, the Pick matrix has the Toeplitz structure

$$P = \begin{bmatrix} w_0 + \bar{w}_0 & \bar{w}_1 & \ldots & \bar{w}_n \\ w_1 & w_0 + \bar{w}_0 & \ldots & \bar{w}_{n-1} \\ \vdots & \vdots & & \vdots \\ w_n & w_{n-1} & \ldots & w_0 + \bar{w}_0 \end{bmatrix},$$

and the problem reduces to the rational covariance extension problem [35, 24, 25, 38, 9, 8, 7, 10, 11].

If the Pick matrix is singular, the solution is unique, rational and of degree $\leq n$, while if $P > 0$ *all* solutions can be described by means of a linear fractional transformation of a "free" parameter function which itself should be positive real [1, 51]. The particular solution obtained by setting the free parameter function equal to one, yields a solution which has degree at most n, thus it also satisfies condition (iii). This is often referred to as the *central* or *maximum entropy* solution. However, the linear fractional transformation is of no help in describing or constructing any other solution to (i)–(iii), because of the complex way in which the free parameter function determines the degree of the interpolant.

The goal of this paper is two-fold. The first is to parameterize all solutions to the Nevanlinna-Pick problem with degree constraint, starting from a generalized notion of entropy for such problems. The second is to provide, through the use of convex optimization, the computational underpinnings for the effective solution of Nevanlinna-Pick problems.

For simplicity, in this paper we shall only consider the case where the interpolation points in $\mathcal{Z}$ are distinct. The general case works similarly. Moreover, we assume that the Pick matrix is positive definite; otherwise there is just one solution. Also, for convenience, we shall normalize the problem so that $z_0 = \infty$ and $f(\infty)$ is real. This is done without loss of generality since, first, the transformation $z \to \frac{1 - \bar{z}_0 z}{z - z_0}$ sends an arbitrary z_0 to infinity and is a bianalytic map from $\mathbb{D}^c$ into itself, and, secondly, we can subtract the same imaginary constant from all values w_k without altering the problem.

Now, condition (ii) requires that

$$f(z) + f^*(z) \geq 0 \quad \text{on the unit circle,}$$

where $f^*(z) := \bar{f}(z^{-1})$. Therefore, if the rational function f is represented as

$$f(z) = \frac{b(z)}{a(z)},$$

where, for the moment, we take $a(z)$ and $b(z)$ to be polynomials, then

$$f(z) + f^*(z) = \frac{\Psi(z)}{a(z)a^*(z)} \tag{2.2}$$

where the pseudo-polynomial Ψ defined by

$$\Psi(z) = a(z)b^*(z) + a^*(z)b(z) \tag{2.3}$$

is nonnegative on the unit circle and hence has a unique stable spectral factor $\sigma(z)$, i.e., a unique polynomial solution of

$$\sigma(z)\sigma^*(z) = \Psi(z)$$

having all its zeros in the closed unit disc $\mathbb{D}$. The functions $a(z)$, $b(z)$ and $\sigma(z)$ can also be rational functions, and this is the formulation that we prefer in this paper. In fact, we shall represent them in a particular space of rational functions having the reciprocals of the points in $\mathbb{Z}$ as their poles. This space will be defined in Section 3.

In this paper we shall device a constructive procedure, akin to that in [10], to show that there is a complete parameterization of all solutions of the Nevanlinna-Pick problem with degree constraint in terms of the zeros of σ, so that for any choice of n zeros in $\mathbb{D}$, there is one and only one solution f. This constructive procedure will provide us with an algorithm to determine the unique solution corresponding to *any* choice of zeros. The zeros can thus be used as the free parameters, and hence as a design tool.

This problem has a long history. In [26], it was shown, for any point sequence $\mathbb{Z}$ and any value sequence $\mathbb{W}$ satisfying the Pick condition, that to each choice of Ψ there corresponded at least one $a(z)$ so that $f = \frac{b}{a}$ is a solution to the Nevanlinna-Pick problem with degree constraint, and hence that the map G from the space of solutions of this problem to the space of monic Schur polynomials, sending f to σ, is surjective. In [26], the question is raised as to whether G is also injective, so that the solutions would be completely parameterized by the choice of zeros of σ. The proof of existence was by means of degree theory and hence nonconstructive. It followed closely the arguments used in [25] to solve an important special case, the rational covariance extension problem. In this setting, σ coincides with the choice of numerator in a shaping filter which will shape white noise into a stationary process with the given covariance sequence. The assertion that G is surjective is then the assertion that the choice of a numerator of such a shaping filter can be made arbitrarily, while the injectivity of G, conjectured in [25], would assert that the choice of numerator completely determines the choice of denominator in the shaping filter.

This conjecture, for the rational covariance extension problem, was recently established in a stronger form in [9], where it is shown that solutions are unique and depend analytically on the problem data. In other words, the rational covariance extension problem is well-posed as an analytic problem. Subsequently, a simpler proof of bijectivity of the parameterization by real Schur polynomials was given in [11] in a form which has been adapted to the rational Nevanlinna-Pick problem in [27], proving that

6   C. I. BYRNES, T. T. GEORGIOU, AND A. LINDQUIST

G indeed provides a complete parameterization of all solutions in terms of the choice of zeros. However, as already mentioned, the proofs developed in [25, 26, 9, 11, 27] are nonconstructive and the question of effectively computing such solutions has remained open. For the rational covariance extension problem, this has been recently addressed in [10] through the development of a convex minimization problem. While the optimization problem of [10] occurs as a special case when specialized to the rational covariance extension problem, our approach to the rational Nevanlinna-Pick problem differs from that in [10] in that we derive a convex optimization scheme as a consequence of a generalized maximum entropy criterion involving the data of the interpolation problem. This criterion is developed in the Sections 4 and 5, where we shall describe a method for computing these solutions, as well as for providing a new, and simple, proof of the parameterization theorem for the rational Nevanlinna-Pick problem using convex optimization methods.

3. Notation and preliminaries

Denote by $\mathcal{L}_2$ the space of functions which are square-integrable on the unit circle. This is a Hilbert space with inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) g^*(e^{i\theta}) d\theta.$$

Moreover, for an $f \in \mathcal{L}_2$, let

$$f(e^{i\theta}) = \sum_{k=-\infty}^{\infty} f_k e^{-ik\theta}$$

be its Fourier representation. In this notation,

$$\langle f, g \rangle = \sum_{k=-\infty}^{\infty} f_k \bar{g}_k.$$

Next, let $\mathcal{H}_2$ be the standard Hardy space of all functions which are analytic in the *exterior* of the unit disc, $\mathbb{D}^c$, and have square-integrable limits on the boundary

$$\lim_{r \to +1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta < \infty.$$

As usual, $\mathcal{H}_2$ is identified with the subspace of $\mathcal{L}_2$ with vanishing negative Fourier coefficients. More precisely, for $f \in \mathcal{H}_2$,

$$f(z) = f_0 + f_1 z^{-1} + f_2 z^{-2} + \dots.$$

Now, consider the data $\mathcal{Z}$ and $\mathcal{W}$ with the standing assumption that $z_0 = \infty$. It is a well-known consequence Beurling's Theorem [32] that the kernel of the evaluation map $E : \mathcal{H}_2 \to \mathbb{C}^{n+1}$ defined via

$$E(f) = \begin{bmatrix} f(z_0) \\ f(z_1) \\ \vdots \\ f(z_n) \end{bmatrix},$$

A GENERALIZED ENTROPY CRITERION FOR RATIONAL INTERPOLATION   7

is given by

$$\ker(E) = B\mathcal{H}_2,$$

where $B(z)$ is the *Blaschke product*

$$B(z) := z^{-1} \prod_{k=1}^{n} \frac{z - z_k}{1 - \bar{z}_k z}.$$

Now, let $\mathcal{H}(B)$ be the orthogonal complement of $B\mathcal{H}_2$ in $\mathcal{H}_2$, i.e., the subspace satisfying

$$\mathcal{H}_2 = B\mathcal{H}_2 \oplus \mathcal{H}(B),$$

which will be referred to as the coinvariant subspace corresponding to B, since $B\mathcal{H}_2$ is invariant under the shift z^{-1}. Then $\mathcal{H}(B)$ is spanned by the (conjugate) Cauchy kernels

$$G_k(z) = \frac{1}{1 - \bar{z}_k^{-1} z^{-1}} = 1 + \bar{z}_k^{-1} z^{-1} + \bar{z}_k^{-2} z^{-2} + \dots$$

for $k = 0, 1, \dots, n$. For any $f = \sum_{j=0}^{\infty} f_j z^{-j} \in \mathcal{H}_2$, we have

$$\langle f, G_k \rangle = \sum_{j=0}^{\infty} f_j z_k^{-j} = f(z_k), \tag{3.1}$$

which, of course, is Cauchy's formula. While $G_k, k = 0, 1, \dots, n$, do form a basis in $H(B)$, we prefer to work in a basis, $g_0, g_1, \dots, g_n$, for which $g_0 = G_0 = 1$ is orthogonal to the rest of the base elements. Thus we choose

$$g_k(z) = G_k(z) - 1 = \frac{1}{z\bar{z}_k - 1}, \quad k = 1, 2, \dots, n.$$

For future reference, we list the four identities

$$\begin{aligned}
\langle f, g_0 \rangle &= f(\infty) \\
\langle f, g_k \rangle &= f(z_k) - f(\infty) \quad k = 1, 2, \dots, n \\
\langle f^*, g_0 \rangle &= \overline{f(\infty)} \\
\langle f^*, g_k \rangle &= 0 \quad k = 1, 2, \dots, n,
\end{aligned} \tag{3.2}$$

which hold for all $f \in \mathcal{H}_2$. In fact, they follow readily from (3.1), $\langle f^*, G_k \rangle = \bar{f}(\infty)$ and the corresponding conjugated identities. We also remark that there is a natural basis for $\mathcal{H}_2$ obtained by extending $\{g_0, g_1, \dots, g_n\}$ by choosing

$$g_k(z) = z^{n+1-k} B(z) \quad \text{for } k = n+1, n+2, \dots.$$

Any element $p \in \mathcal{H}(B)$ is of the form

$$p(z) = \frac{\pi(z)}{\tau(z)},$$

where

$$\tau(z) = \prod_{k=1}^{n} (z - \bar{z}_k^{-1}) \tag{3.3}$$

8   C. I. BYRNES, T. T. GEORGIOU, AND A. LINDQUIST

and $\pi(z) = \pi_0 z^n + \pi_1 z^{n-1} + \cdots + \pi_n$ is also a polynomial of degree at most n. Finally, any rational function of degree at most n can be written as

$$f(z) = \frac{b(z)}{a(z)} \quad \text{where } a, b \in \mathcal{H}(B).$$

Throughout the paper we shall use such representations for rational functions, and in particular the functions $a(z)$, $b(z)$ and $\sigma(z)$, introduced in Section 2 will belong to $\mathcal{H}(B)$. Hence, $\Psi(z)$, defined by (2.3) will be a symmetric pseudo-polynomial in the basis elements of $\mathcal{H}(B)$ and $\mathcal{H}(B)^*$. In general, the space of pseudo-polynomials in this basis will be denoted by $\mathcal{S}$, and is defined by

$$\mathcal{S} = \mathcal{H}(B) \vee \mathcal{H}(B)^* = \text{span}\{g_n^*, \ldots, g_1^*, g_0, g_1, \ldots, g_n\}. \tag{3.4}$$

In particular, $\Psi \in \mathcal{S}$, and so are ab^* and a^*b.

4. A generalized entropy criterion for Nevanlinna-Pick interpolation

Given a rational positive real function $f(z)$ we consider the generalized entropy gain

$$\mathbb{I}_\Psi(f) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \log[\Phi(e^{i\theta})]\Psi(e^{i\theta})d\theta \tag{4.1}$$

where $\Psi(z)$ is a specified spectral density function in $\mathcal{S}$ which is positive on the unit circle, and

$$\Phi(z) := f(z) + f^*(z). \tag{4.2}$$

In fact, $\Psi(z)$ can be factored as

$$\Psi(z) = \sigma(z)\sigma^*(z), \tag{4.3}$$

where $\sigma \in \mathcal{H}(B)$ has no zeros in the closure of $\mathbb{D}^c$, i.e., $\sigma(z)$ is a minimum-phase spectral factor of $\Psi(z)$.

Entropy integrals such as (4.1) have, of course, a long history. In particular, one might compare this particular generalized entropy integral with that developed in [42] for $\mathcal{H}^\infty$ control. While Nevanlinna-Pick interpolation is quite relevant in $\mathcal{H}^\infty$ control, the entropy formula (4.1) is defined on $\mathcal{H}^2$ and does not involve the $\mathcal{L}^2$ gain of a system. Indeed, it is closer to the entropy expression used to derive the maximal entropy filters in signal processing (see, e.g., [31, 36]).

We are interested in solutions to the Nevanlinna-Pick problem with degree constraint presented in Section 2. It turns out that there is a unique solution $f(z)$ which maximizes the above entropy functional. Moreover, this solution satisfies

$$f(z) + f^*(z) = \frac{\sigma(z)\sigma^*(z)}{a(z)a^*(z)}, \tag{4.4}$$

where $\sigma, a \in \mathcal{H}(B)$ and $\sigma\sigma^* = \Psi$. Hence the entropy maximization forces a preselected spectral zero structure for the interpolating function. In fact, it will be shown below that this provides a complete parameterization of all such rational solutions of at most degree n.

Theorem 4.1. *Assume that $\mathcal{Z}$ and $\mathcal{W}$ define an interpolation problem for which the Pick matrix P is positive definite. Given a $\Psi \in \mathcal{S}$ which is positive on the unit circle, there exists a unique solution to the constrained optimization problem*

$$\max_{f \in \mathcal{C}_0} \mathbb{I}_\Psi(f) \qquad (4.5)$$

subject to the constraints

$$f(z_k) = w_k \quad \text{for } k = 0, 1, \ldots, n. \qquad (4.6)$$

Moreover, this solution is of the form

$$f(z) = \frac{b(z)}{a(z)}, \quad a, b \in H(B), \qquad (4.7)$$

and hence of degree at most n, and

$$a(z)b^*(z) + b(z)a^*(z) = \Psi(z). \qquad (4.8)$$

Conversely, if $f \in \mathcal{C}_0$ satisfies conditions (4.6), (4.7) and (4.8), it is the unique solution of (4.5).

The proof of this theorem will be deferred to the next section.

In the special case where $\Psi = 1$,

$$\mathbb{I}_1(f) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \log[f(e^{i\theta}) + f^*(e^{i\theta})]d\theta \qquad (4.9)$$

corresponds to the standard entropy criterion which is maximized by the central solution mentioned in Section 2. It is clear that, in general, maximization of $\mathbb{I}_\Psi(f)$ is unaffected by scaling Ψ by any positive constant factor. Theorem 4.1 provides a complete parameterization of all strictly positive real solutions to the Nevanlinna-Pick problem with degree constraint in terms of properly scaled spectral densities $\Psi \in \mathcal{S}$, or, in other words, in terms of the zeros of the minimum-phase spectral density $\sigma(z)$.

Corollary 4.2 (Spectral Zero Assignability Theorem). *There is a bijective correspondence between solutions $f \in \mathcal{C}_0$ to the Nevanlinna-Pick problem with degree constraint and the set*

$$\{\Psi \in \mathcal{S} \mid \Psi(e^{i\theta}) > 0 \text{ for all } \theta; \ \frac{1}{2\pi} \int_{-\pi}^{\pi} \Psi(e^{i\theta})d\theta = 1\},$$

or, equivalently, the set of n points in the open unit disc, these being the zeros of the minimum-phase spectral factor $\sigma(z)$.

Problem (4.5) is an infinite-dimensional optimization problem. However, since there are only finitely many interpolation constraints, there is a dual problem with finitely many variables.

Returning to conditions (4.7) and (4.8), we see that

$$f(z) + f^*(z) = \frac{\Psi(z)}{Q(z)},$$

10   C. I. BYRNES, T. T. GEORGIOU, AND A. LINDQUIST

where $Q(z) = a(z)a^*(z) \in \mathbb{S}$, and $Q(z) > 0$ on the unit circle. In terms of the basis introduced in Section 2,

$$Q(z) = \bar{q}_n g_n^*(z) + \ldots + \bar{q}_1 g_1^*(z) + q_0 g_0(z) + q_1 g_1(z) + \ldots + q_n g_n(z). \quad (4.10)$$

Since $g_0(z) \equiv 1$, $q_0 = \langle Q, g_0 \rangle = \int_{-\pi}^{\pi} Q(e^{i\theta})d\theta$. Therefore, since Q is positive on the circle, q_0 is real and positive. Hence, we may identify Q with the vector $q :=$ $(q_0, q_1, \ldots, q_n)$ of coefficients belonging to the set

$$\mathfrak{Q} := \{q \in \mathbb{R} \times \mathbb{C}^n \mid Q(e^{i\theta}) > 0 \quad \text{for all } \theta\}.$$

As we shall see shortly the q-parameters will essentially be the Lagrange multipliers for the dual problem.

Consider the Lagrange function

$$L(f, \lambda) = \mathbb{I}_{\Psi}(f) + \lambda_0(w_0 - f(z_0)) + 2\mathrm{Re}\left\{\sum_{k=1}^{n} \bar{\lambda}_k[w_k - f(z_k)]\right\}. \quad (4.11)$$

Since the primal problem amounts to maximizing a strictly concave function over a convex region, the Lagrange function has a saddle point [41, p.458] provided there is a stationary point in $\mathbb{C}_0$, and, in this case, the optimal Lagrange vector $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n) \in \mathbb{C}^{n+1}$ can be determined by solving the *dual problem* to minimize

$$\rho(\lambda) = \max_{f \in \mathbb{C}_0} L(f, \lambda). \quad (4.12)$$

Proposition 4.3. *Given a vector* $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n)$ *of Lagrange multipliers, the unique maximizing function* f *in problem* (4.12) *satisfies*

$$f(z) + f^*(z) = \frac{\Psi(z)}{Q(z)},$$

where the coefficients of Q *are related to the Lagrange multipliers as follows:*

$$q_0 = \tfrac{1}{2}\lambda_0 + Re\sum_{j=1}^{n} \lambda_j$$

$$q_k = \lambda_k \quad for \ k = 1, 2, \ldots, n.$$

Proof. We note that $\mathbb{C}_0 \subset \mathcal{H}^2$, and we consider the representation

$$f(z) = \sum_{j=0}^{\infty} f_j g_j(z). \quad (4.13)$$

Based on our standing assumptions on $f(z)$, and our choice of the basis $\{g_k(z); k = 0, 1, 2, \ldots\}$, we have $f_0 = f(\infty)$ is real, while f_k, $k = 1, 2, \ldots$, are allowed to be complex. Thus, we identify $f(z)$ with the vector of coefficients $f := (f_0, f_1, \ldots)$, and define the set

$$\mathcal{F} = \{f \in \ell_2 \mid f_0 \in \mathbb{R}; \ f_1, f_2, \cdots \in \mathbb{C}; \ \sum_{j=0}^{\infty} f_j g_j(z) \in \mathbb{C}_0\}. \quad (4.14)$$

A GENERALIZED ENTROPY CRITERION FOR RATIONAL INTERPOLATION   11

Since $B(z_k) = 0$ for $k = 0, 1, 2, \ldots, n$, we have $g_j(z_k) = 0$ for $j > n$, and consequently

$$f(z_k) = \sum_{j=0}^{n} f_j g_j(z_k). \tag{4.15}$$

In what follows, it will be convenient to use complex partial differential operators acting on smooth, but not necessarily complex analytic, functions. In particular, if we write the complex vector $f_k = x_k + iy_k$ as a sum of real and imaginary parts, this defines the differential operators

$$\frac{\partial}{\partial f_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{f}_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k}\right)$$

which operate on smooth functions. Indeed, the second operator is the Cauchy-Riemann operator which characterizes the analytic functions F of f_k via

$$\frac{\partial F}{\partial \bar{f}_k} = 0.$$

And, for example, while conjugation, viewed as the function defined by $\bar{f}_k = x_k - iy_k$, is of course not analytic, it is smooth and satisfies

$$\frac{\partial \bar{f}_k}{\partial f_k} = 0 \quad \text{and} \quad \frac{\partial \bar{f}_k}{\partial \bar{f}_k} = 1.$$

Returning to the maximization problem (4.12), we set $\frac{\partial L}{\partial f_k} = 0$ for all k. Since f_0 is real and $g_0 = 1$, we then have

$$\frac{\partial L}{\partial f_0} = 2\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi^{-1}(e^{i\theta})\Psi(e^{i\theta})d\theta - \lambda_0 - 2\mathrm{Re}\left\{\sum_{k=1}^{n}\bar{\lambda}_k\right\} = 0. \tag{4.16}$$

Furthermore, recalling (4.15), we obtain

$$\frac{\partial L}{\partial f_k} = \frac{1}{2\pi}\int_{-\pi}^{\pi}g_k(e^{i\theta})\Phi^{-1}(e^{i\theta})\Psi(e^{i\theta})d\theta - \sum_{j=1}^{n}\bar{\lambda}_j g_k(z_j) = 0 \tag{4.17}$$

for $k = 1, 2, \ldots, n$, and

$$\frac{\partial L}{\partial f_k} = \frac{1}{2\pi}\int_{-\pi}^{\pi}g_k(e^{i\theta})\Phi^{-1}(e^{i\theta})\Psi(e^{i\theta})d\theta = 0 \tag{4.18}$$

for $k = n + 1, n + 2, \ldots$. Now, let $Q(z) := \Phi^{-1}(z)\Psi(z)$, and note that $Q^*(z) = Q(z)$. From (4.18),

$$\langle Q, g_k \rangle = 0 = \langle Q, g_k^* \rangle \quad \text{for } k = n + 1, n + 2, \ldots.$$

Hence $Q \in \mathcal{S}$. Therefore there is a representation (4.10) with $q_0 \in \mathbb{R}$ and $q_1, \ldots, q_n \in \mathbb{C}$. Moreover, $Q(e^{i\theta}) > 0$ for all θ.

From (4.16), we immediately see that

$$\lambda_0 = 2q_0 - 2\mathrm{Re}\left\{\sum_{j=1}^{n}\lambda_j\right\}. \tag{4.19}$$

Next, taking the conjugate of (4.17) we obtain

$$\langle Q, g_k \rangle = \sum_{j=1}^{n} \lambda_j \bar{g}_k(\bar{z}_j) \tag{4.20}$$

for $k = 1, 2, \ldots, n$. On the other hand,

$$\langle Q, g_k \rangle = \sum_{j=1}^{n} q_j g_j(z_k). \tag{4.21}$$

Since $\bar{g}_k(\bar{z}_j) = g_j(z_k)$, by (4.20) and (4.21),

$$\begin{bmatrix} g_1(z_1) & g_2(z_1) & \cdots & g_n(z_1) \\ g_1(z_2) & g_2(z_2) & \cdots & g_n(z_2) \\ \vdots & \vdots & \ddots & \vdots \\ g_1(z_n) & g_2(z_n) & \cdots & g_n(z_n) \end{bmatrix} \begin{bmatrix} \lambda_1 - q_1 \\ \lambda_2 - q_2 \\ \vdots \\ \lambda_n - q_n \end{bmatrix} = 0. \tag{4.22}$$

Now, it is easy to see that the coefficient matrix

$$G = \left[\frac{\bar{z}_k^{-1} z_\ell^{-1}}{1 - \bar{z}_k^{-1} z_\ell^{-1}} \right]_{k,\ell=1}^{n}, \tag{4.23}$$

of the linear system (4.22) is nonsingular, and therefore

$$\lambda_k = q_k \quad \text{for } k = 1, 2, \ldots, n. \tag{4.24}$$

In fact, $G = \frac{1}{2} Z^* P Z$, where Z is the diagonal matrix $\text{diag}(z_1^{-1}, \ldots, z_n^{-1})$ and P is the Pick matrix for $\mathcal{Z} = \{z_1, \ldots, z_n\}$ and $\mathcal{W} = \{1, \ldots, 1\}$. Consequently, since this interpolation problem has infinitely many solutions, by the Pick condition, P must be positive definite. This concludes the proof of the proposition. $\square$

It turns out to be more convenient to use the q's as dual variables.

Proposition 4.4. *The dual functional (4.12) is*

$$\rho(\lambda(q)) = \varphi(q) + c,$$

where c is a constant, and

$$\varphi(q) = 2w_0 q_0 + 2Re\left\{ \sum_{k=1}^{n} (w_k - w_0)\bar{q}_k \right\} - \frac{1}{2\pi} \int_{-\pi}^{\pi} \log[Q(e^{i\theta})]\Psi(e^{i\theta})d\theta \tag{4.25}$$

Proof. As we have just seen, the Lagrange multipliers are linear functions of the $q := (q_0, q_1, \ldots, q_n)$. The dual functional (4.12) becomes

$$\rho(\lambda(q)) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \log[Q(e^{i\theta})]\Psi(e^{i\theta})d\theta + \frac{1}{2\pi} \int_{-\pi}^{\pi} \log[\Psi(e^{i\theta})]\Psi(e^{i\theta})d\theta$$

$$+ \left(2q_0 - 2Re\left\{ \sum_{j=1}^{n} q_j \right\} \right) (w_0 - f_0) + 2Re\left\{ \sum_{j=1}^{n} \bar{q}_j[w_j - f(z_j)] \right\}. \tag{4.26}$$

In this expression the sum of the two last terms turns out to be linear in q. To see this and eliminate the dependence of f's on the q's, consider the following:

$$
\frac{1}{2\pi}\int_{-\pi}^{\pi}\Psi(e^{i\theta})d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}Q(e^{i\theta})\Phi(e^{i\theta})d\theta
$$

$$
= q_0\langle f+f^*, g_0\rangle + 2\mathrm{Re}\left\{\sum_{j=1}^{n}\bar{q}_j\langle f+f^*, g_j\rangle\right\}
$$

$$
= 2q_0 f_0 + 2\mathrm{Re}\left\{\sum_{j=1}^{n}\bar{q}_j(f(z_j)-f_0)\right\}.
$$

Using this last expression, the dual function becomes

$$
\rho(\lambda(q)) = -\frac{1}{2\pi}\int_{-\pi}^{\pi}\log[Q(e^{i\theta})]\Psi(e^{i\theta})d\theta + \frac{1}{2\pi}\int_{-\pi}^{\pi}\log[\Psi(e^{i\theta})]\Psi(e^{i\theta})d\theta
$$

$$
-\frac{1}{2\pi}\int_{-\pi}^{\pi}\Psi(e^{i\theta})d\theta + 2q_0 w_0 + 2\mathrm{Re}\left\{\sum_{j=1}^{n}\bar{q}_j(w_j-w_0)\right\}. \tag{4.27}
$$

In this expression, define c to be the sum of the second and third terms. Then, the proposition follows. $\square$

We are now in a position to formulate the dual version of Theorem 4.1, the proof of which will also be deferred to the next section.

Theorem 4.5. *Assume that $\mathcal{Z}$ and $\mathcal{W}$ define an interpolation problem for which the Pick matrix P is positive definite. Given a $\Psi \in \mathcal{S}$ which is positive on the unit circle, there exists a unique solution to the dual problem*

$$
\min_{q\in\Omega}\varphi(q). \tag{4.28}
$$

Moreover, for the minimizing q,

$$
\frac{\Psi(z)}{Q(z)} = f(z) + f^*(z) \tag{4.29}
$$

with $f \in \mathcal{C}_0$. Moreover, this function f satisfies conditions (4.6), (4.7) and (4.8) in Theorem 4.1, namely

$$
f(z_k) = w_k \quad \text{for } k = 0, 1, \dots, n, \tag{4.30}
$$

$$
f(z) = \frac{b(z)}{a(z)}, \quad a, b \in H(B), \tag{4.31}
$$

$$
\Psi(z) = a(z)b^*(z) + b(z)a^*(z). \tag{4.32}
$$

Conversely, any $f \in \mathcal{C}_0$ which satisfies these conditions can be constructed from the unique solution of (4.28) via (4.29).

We conclude by noting that if the problem data is real or self-conjugate, and Ψ is real, then both the function $f(z)$ constructed above, and the function $\bar{f}(\bar{z})$, satisfy the conditions of Theorems 4.1 and 4.5 so that, by uniqueness, they must coincide.

Corollary 4.6. *Assume that the the sets $\mathcal{Z}$ and $\mathcal{W}$ are self-conjugate and that $w_k = \bar{w}_j$ whenever $z_k = \bar{z}_j$, and that Ψ is real. Then, the optimizing functions f, Q in Theorems 4.1 and 4.5 have real coefficients. In particular, there is a unique pair of real functions $a(z)$ and $b(z)$ in $\mathcal{H}(B)$, devoid of zeros in closure of $\mathbb{D}^c$, such that*

$$\Psi(z) = a(z)b(z^{-1}) + a(z^{-1})b(z)$$

$$f(z) = \frac{b(z)}{a(z)} \in \mathcal{C}_0$$

$$f(z_k) = w_k \quad for \ k = 0, 1, ..., n.$$

We shall return to the special case covered in Corollary 4.6 in Section 7, and we shall refer to it as the *self-conjugate case*.

5. The convex optimization problem

In this section, we shall analyze the functional $\varphi(q)$, constructed in the previous section. We shall show that it has a unique minimum in $\mathcal{Q}$, and this will be instrumental in proving Theorem 4.1 and Theorem 4.5, which will be done at the end of the section. To this end, we shall extend $\varphi(q)$ to the closure $\overline{\mathcal{Q}}$ of $\mathcal{Q}$, and consider

$$\varphi : \overline{\mathcal{Q}} \longrightarrow \mathbb{R} \cup \{\infty\}.$$

Proposition 5.1. *The functional $\varphi(q)$ is a C^∞ function on $\mathcal{Q}$ and has a continuous extension to the boundary that is finite for all $q \neq 0$. Moreover, φ is strictly convex, and $\overline{\mathcal{Q}}$ is a closed and convex set.*

This proposition, along with Propositions 5.2 and 5.4 below, are analogous to related results in [10], developed for the covariance extension problem. Their proofs are similar, *mutatis mutandis*, to those developed in [10], except for Lemma 5.3 below. The complete proofs are adapted to the present framework and included in the appendix for the convenience of the reader.

In order to ensure that φ achieves a minimum on $\overline{\mathcal{Q}}$, it is important to know whether φ is proper, i.e., whether $\varphi^{-1}(K)$ is compact whenever K is compact. In this case, of course, a unique minimum will exist.

Proposition 5.2. *For all $r \in \mathbb{R}$, $\varphi^{-1}(-\infty, r]$ is compact. Thus φ is proper (i.e., $\varphi^{-1}(K)$ is compact whenever K is compact) and bounded from below.*

The proof of this proposition, given in the appendix, relies on the analysis of the growth of φ, which entails a comparison of linear and logarithmic growth. To this end, the following lemma is especially important. We note that its proof is the only point in our construction and argument in which we use the Pick condition in an essential way.

A GENERALIZED ENTROPY CRITERION FOR RATIONAL INTERPOLATION   15

Lemma 5.3. *Suppose that the Pick matrix P is positive definite. Let $\varphi_1(q)$ be the linear part of $\varphi(q)$, i.e.,*

$$\varphi_1(q) := 2w_0 q_0 + 2Re\left\{ \sum_{k=1}^{n}(w_k - w_0)\bar{q}_k \right\}$$

$$= 2w_0 q_0 + \sum_{k=1}^{n}(w_k - w_0)\bar{q}_k + \sum_{k=1}^{n}(\bar{w}_k - w_0)q_k. \tag{5.1}$$

Then $\varphi_1(q) > 0$ for each nonzero $q \in \overline{\mathfrak{Q}}$.

Proof. Since $P > 0$, there exists a strictly positive real interpolant of the interpolation sequence $(z_k, w_k); k = 0, 1, \ldots, n$. Choose an arbitrary such interpolant, and denote it by f. Then, recalling that $z_0 = \infty$, (3.2) yields

$$2w_0 = \langle f + f^*, g_0 \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(e^{i\theta}) + f^*(e^{i\theta})]g_0^*(e^{i\theta})d\theta$$

and

$$w_k - w_0 = \langle f + f^*, g_k \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(e^{i\theta}) + f^*(e^{i\theta})]g_k^*(e^{i\theta})d\theta$$

for $k = 1, 2, \ldots, n$. For any q in $\overline{\mathfrak{Q}}_n$, we compute

$$\varphi_1(q) = \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(e^{i\theta}) + f^*(e^{i\theta})]Q(e^{i\theta})d\theta \geq 0,$$

and $\varphi_1(q) = 0$ if and only if $Q \equiv 0$. $\square$

Finally, we need to exclude the possibility that the minimum occurs on the boundary. This is the content of the following proposition, also proved in the appendix.

Proposition 5.4. *If Ψ is positive on the unit circle, the functional φ never attains a minimum on the boundary $\partial\mathfrak{Q}$.*

Hence we have established that $\varphi(q)$ is strictly convex, has compact sublevel sets and the minimum does not occur on the boundary of $\mathfrak{Q}$. Consequently, it has a unique minimum, which occurs in the open set $\mathfrak{Q}$. Clearly, this minimum point will be a stationary point with vanishing gradient. As the following lemma shows, the gradient becomes zero precisely when the interpolation conditions are satisfied, and in fact the value of the gradient depends only on the mismatch at the interpolation points.

Lemma 5.5. *At any point $q \in \mathfrak{Q}$ the gradient of φ is given by*

$$\frac{\partial\varphi}{\partial q_0} = 2[w_0 - f(z_0)], \tag{5.2}$$

$$\frac{\partial\varphi}{\partial\bar{q}_k} = [w_k - f(z_k)] - [w_0 - f(z_0)], \quad for\ k = 1, 2, \ldots, n, \tag{5.3}$$

where f is the $\mathbb{C}_0$ function satisfying

$$f(z) + f^*(z) = \frac{\Psi(z)}{Q(z)} \tag{5.4}$$

16   C. I. BYRNES, T. T. GEORGIOU, AND A. LINDQUIST

with $Q(z) \in \mathcal{S}$ *correspond to* q *as in* (4.10).

Proof. The existence of a function f as claimed in the statement is obvious by virtue of the fact that $\frac{\Psi(z)}{Q(z)}$ is bounded and greater than zero on the unit circle. Recalling that

$$\frac{\partial q_k}{\partial \bar{q}_k} = 0,$$

for $k > 0$ we have

$$\frac{\partial \varphi}{\partial \bar{q}_k} = (w_k - w_0) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{g_k^*(e^{i\theta})}{Q(e^{i\theta})} \Psi(e^{i\theta}) d\theta$$
$$= (w_k - w_0) - \langle f + f^*, g_k \rangle,$$

which, in view of (3.2) and the fact that $z_0 = \infty$, is the same as (5.3). For the case $k = 0$ we need to take the real derivative:

$$\frac{\partial \varphi}{\partial q_0} = 2w_0 - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{g_0(e^{i\theta})}{Q(e^{i\theta})} \Psi(e^{i\theta}) d\theta$$
$$= 2w_0 - \langle f + f^*, g_0 \rangle,$$

which, again using (3.2), yields (5.2). $\square$

We are now prepared for the proof of our main results.

Proof of Theorem 4.5. Propositions 5.1, 5.2 and 5.4 establish the existence of a unique minimum in $q \in \mathcal{Q}$. Then Lemma 5.5 shows that the interpolation conditions are met for the corresponding $\mathcal{C}_0$-function f satisfying (5.4). The construction of such a function proceeds as follows. Since $Q \in \mathcal{S}$ and is positive and bounded on the unit circle, it admits a rational spectral factorization $Q(z) = a(z)a^*(z)$, where $a(z) = \frac{\alpha(z)}{\tau(z)}$ with $\alpha(z)$ a stable polynomial of degree at most n. Hence, $a \in \mathcal{H}(B)$. Then, we solve the linear equation $a(z)b^*(z) + b(z)a^*(z) = \Psi(z)$ for b. This linear equation has always a unique solution because a has no roots in $\mathbb{D}^c$; cf. the discussion in [12]. Then $f(z) = \frac{b(z)}{a(z)}$, and all conditions of the theorem are satisfied.

Conversely, given an $f \in \mathcal{C}_0$ satisfying (4.31) and (4.32), a unique $q \in \mathcal{Q}$ can be obtained from (5.4). Finally, in view of Lemma 5.5, the interpolation conditions (4.30) imply that the gradient of φ for the corresponding q is zero. Thus it is the unique minimizing q. $\square$

Proof of Theorem 4.1. The fact that the minimizing q in Theorem 4.5 belongs $\mathcal{Q}$ is equivalent to having the corresponding f in $\mathcal{C}_0$. Since both $\mathcal{Q}$ and $\mathcal{C}_0$ are open they do not impose binding constrains on the primal and dual problems. Hence, by standard duality theory [41, p. 458], the Lagrangian (4.11) has a saddle point. Consequently, the is a direct correspondence between the primal and dual problems which translates the statements of Theorem 4.5 to the corresponding ones of Theorem 4.1. $\square$

6. Computational procedure

An interesting, and useful, aspect of the dual functional $\varphi(q)$ is that it contains a barrier-like term, as used in interior point methods. However, as we have seen in Section 5, by a theorem of Szegö, the logarithmic integrand is in fact integrable for nonzero Q having zeros on the boundary of the unit circle, and hence $\varphi(q)$ does not become infinite on the entire boundary $\partial\Omega$ of Ω. For this reason it is not a barrier term in the traditional sense. Nonetheless, $\varphi(q)$ has the very interesting barrier-type property described in the following proposition, which is a simple corollary to Proposition 5.1 and Proposition 5.4.

Proposition 6.1. *Although the dual functional $\varphi(q)$ is finite in each nonzero point on the boundary $\partial\Omega$, it has an infinite gradient there.*

Next, let us turn to the computational procedure. Given $\Psi(z)$, define the class $\mathcal{P}$ of (strictly) positive real functions

$$f(z) = \frac{b(z)}{a(z)}, \quad a, b \in \mathcal{H}(B)$$

having the property that

$$a(z)b^*(z) + b(z)a^*(z) = \Psi(z). \tag{6.1}$$

We want to determine the unique function in $\mathcal{P}$ which also satisfies the interpolation conditions. To this end, we shall construct a sequence of functions,

$$f^{(0)}, f^{(1)}, f^{(2)}, \cdots \in \mathcal{P}$$

which converges to this interpolant.

As before, we may write (6.1) as

$$f(z) + f^*(z) = \frac{\Psi(z)}{Q(z)}, \tag{6.2}$$

where $Q \in \mathcal{S}$ satisfies

$$a(z)a^*(z) = Q(z). \tag{6.3}$$

It is easy to see that this defines a bijection

$$\mathcal{J} : \Omega \to \mathcal{P} : \quad Q \mapsto f. \tag{6.4}$$

To see this, note that

$$a(z) = \frac{\alpha(z)}{\tau(z)}, \quad b(z) = \frac{\beta(z)}{\tau(z)} \quad \text{and} \quad \Psi(z) = \frac{d(z, z^{-1})}{\tau(z)\tau^*(z)},$$

where $\alpha(z)$ and $\beta(z)$ are Schur polynomials of at most degree n and $d(z, z^{-1})$ is a pseudo-polynomial, also of at most degree n. Then, determine $\alpha(z)$ via a stable polynomial factorization

$$\alpha(z)\alpha^*(z) = \tau(z)\tau^*(z)Q(z), \tag{6.5}$$

and solve the linear system

$$\alpha(z)\beta^*(z) + \beta(z)\alpha(z)^* = d(z, z^{-1}) \tag{6.6}$$

for β. In fact, (6.6) is a linear (Hankel + Toeplitz) system $S(\alpha)\beta = d$ in the coefficients of the polynomials, which is nonsingular since $\alpha(z)$ is a Schur polynomial; see, e.g., [12]. Then

$$f(z) = \frac{\beta(z)}{\alpha(z)}.$$

Given an $f \in \mathcal{P}$ we can determine the corresponding gradient of $\varphi(q)$ by means of Lemma 5.5. The following lemma gives the equations for the $(n+1) \times (n+1)$ Hessian matrix

$$H(q) = \left[\frac{\partial^2 \varphi}{\partial \bar{q}_k \partial \bar{q}_\ell} \right]_{k,\ell=0}^{n}. \tag{6.7}$$

Lemma 6.2. *Let $h(z)$ be the unique positive real function such that*

$$h(z) + h^*(z) = \frac{\Psi(z)}{Q(z)^2} \tag{6.8}$$

and $h(z_0)$ is real. Then the Hessian (6.7) is given by

$$H_{k\ell}(q) = \begin{cases} \frac{z_k}{z_\ell - z_k} h(z_k) + \frac{z_\ell}{z_k - z_\ell} h(z_\ell) + h(z_0) & \text{for } k \neq \ell; k, \ell > 0 \\ -z_k h'(z_k) - h(z_k) + h(z_0) & \text{for } k = \ell > 0 \\ h(z_k) - h(z_0) & \text{for } k > 0, \ell = 0 \\ h(z_\ell) - h(z_0) & \text{for } k = 0, \ell > 0 \\ 2h(z_0) & \text{for } k = \ell = 0, \end{cases} \tag{6.9}$$

where $h'(z)$ is the derivative of $h(z)$.

Proof. For $k, \ell = 0, 1, \ldots, n$ we have

$$\frac{\partial^2 \varphi}{\partial \bar{q}_k \partial \bar{q}_\ell} = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_k^*(e^{i\theta}) g_\ell^*(e^{i\theta}) \frac{\Psi(e^{i\theta})}{Q(e^{i\theta})^2} d\theta \tag{6.10}$$

$$= \langle (h + h^*) g_\ell^*, g_k \rangle. \tag{6.11}$$

For $\ell = 0$ this becomes $\langle h, g_k \rangle + \langle h^*, g_k \rangle$, which, in view of (3.2), becomes $h(z_k) - h(z_0)$ if $k > 0$ and $2h(z_0)$ if $k = 0$. For $k, \ell > 0$, we have $\langle h^* g_\ell^*, g_k \rangle = 0$ and therefore

$$\frac{\partial^2 \varphi}{\partial \bar{q}_k \partial \bar{q}_\ell} = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_k^*(e^{i\theta}) g_\ell^*(e^{i\theta}) h(e^{i\theta}) d\theta.$$

There are two cases. First, suppose $k \neq \ell$. Then a simple calculation yields

$$g_k^*(z) g_\ell^*(z) = \frac{z_k}{z_\ell - z_k} g_k^*(z) + \frac{z_\ell}{z_k - z_\ell} g_\ell^*(z),$$

and hence

$$\frac{\partial^2 \varphi}{\partial \bar{q}_k \partial \bar{q}_\ell} = \frac{z_k}{z_\ell - z_k} \langle h, g_k \rangle + \frac{z_\ell}{z_k - z_\ell} \langle h, g_\ell \rangle,$$

which, by (3.2), yields those elements of the Hessian for which $k \neq \ell$ and $k, \ell > 0$. Secondly, suppose that $k = \ell$. Since

$$g_k^*(z) = \frac{z}{z_k - z} = -1 + \frac{z_k}{z_k - z},$$

A GENERALIZED ENTROPY CRITERION FOR RATIONAL INTERPOLATION   19

we obtain

$$\frac{\partial^2 \varphi}{\partial^2 \bar{q}_k} = -\langle h, g_k \rangle + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{z_k e^{i\theta}}{(z_k - e^{i\theta})^2} h(e^{i\theta}) d\theta. \qquad (6.12)$$

To compute the second term in (6.12), differentiate $h(z)$, which is given, as above, by the Cauchy formula

$$h(z) - h(z_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta}}{z - e^{i\theta}} h(e^{i\theta}) d\theta.$$

Then

$$h'(z) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta}}{(z - e^{i\theta})^2} h(e^{i\theta}) d\theta,$$

which, together with (6.12) and (3.2), proves the remaining part of the lemma. □

Next, we turn to the computational procedure, which will be based on Newton's method [40, 41]. We need an $f^{(0)} \in \mathcal{P}$, and a corresponding $Q^{(0)}$ determined via (6.2), as an initial condition. For this we may choose the "central solution" (see Section 4) for which there is a simple algorithm. Each iteration in our procedure consists of four steps and updates the pair f, Q to $\hat{f}, \hat{Q}$, in the following way:

Step 1. Given f, let $\nabla\varphi(q)$ be the gradient defined by (5.2) and (5.3).

Step 2. Determine the unique positive real function h satisfying (6.8), which is a linear problem of the same type as the one used to determine f from Q. In fact, exchanging $\alpha(z)$ for $\alpha(z)^2$ and $d(z, z^{-1})$ for $v(z, z^{-1}) = \tau(z)\tau^*(z)d(z, z^{-1})$ in (6.6) we obtain

$$h(z) = \frac{\beta(z)}{\alpha(z)} \quad \text{where } \beta = S(\alpha^2)v.$$

The Hessian $H(q)$ is then determined from h as in Lemma 6.2.

Step 3. Update $Q(z)$ by applying Newton's method to the function φ. A Newton step yields

$$\hat{q} = q - \lambda H(q)^{-1} \nabla\varphi(q),$$

where $\lambda \in (0, 1]$ needs to chosen so that

$$\hat{Q}(e^{i\theta}) > 0 \quad \text{for all } \theta. \qquad (6.13)$$

This positivity condition is tested in Step 4.

Step 4. Factor $\hat{Q}$ as in (6.3). This is also a test for condition (6.13). If the test fails, return to Step 3 and decrease the step size λ. Otherwise, use the linear procedure above to determine the next iterate $\hat{f}$. Check if $\hat{f}$ is sufficiently close to f. If so, stop; otherwise, set $f := \hat{f}$ and return to Step 1.

7. State space formulas

The computations of the previous section can be carried out quite efficiently using state space descriptions. In this section we return to the self-conjugate case, where both $\mathcal{Z}$ and $\mathcal{W}$ are self-conjugate and $w_k = \bar{w}_j$ whenever $z_k = \bar{z}_j$, and $\Psi(z)$ is real. (See Corollary 4.6.) In particular, we shall develop the steps of the algorithm so as to avoid complex arithmetic.

It is easy to see that, in this case,

$$\tau(z) := \prod_{k=1}^{n}(z - \bar{z}_k^{-1}) = z^n + \tau_1 z^{n-1} + \cdots + \tau_n \tag{7.1}$$

is a real polynomial and

$$B(z) = z^{-1}\frac{\tau_*(z)}{\tau(z)} \tag{7.2}$$

is a real function, where $\tau_*(z) := 1 + \tau_1 z + \cdots + \tau_n z^n$ is the reverse polynomial. Throughout this section, we shall be concerned with real interpolation functions.

Any real function $h \in \mathcal{H}(B)$ admits a state space representation of the form

$$h(z) = h_0 + c(zI - A)^{-1}h, \tag{7.3}$$

where (A, h, c) are taken in the observer canonical form

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -\tau_n & -\tau_{n-1} & \cdots & -\tau_1 \end{bmatrix} \qquad h = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix} \tag{7.4}$$

$$c = \begin{bmatrix} 1, 0, \ldots, 0 \end{bmatrix},$$

$h_1, h_2, \ldots, h_n$ being the Markov parameters in the Taylor expansion

$$h(z) = h_0 + h_1 z^{-1} + \cdots + h_n z^{-n} + \ldots$$

about infinity. We shall use the compact notation

$$\mathbf{h} = \left[\begin{array}{c|c} A & h \\ \hline c & h_0 \end{array}\right]$$

for this representation, and keep A and c fixed when representing real functions in $\mathcal{H}(B)$. Since the function (7.3) is completely determined by the Markov parameters h_0, h, we shall refer to them as the *Markov coordinates* of the function (7.3). Alternatively, $h(z)$ can also be represented with respect to the standard basis in $\mathcal{H}(B)$ as

$$h(z) = h_0 + \sum_{j=1}^{n} \eta_j g_j(z), \tag{7.5}$$

where, of course, $\eta_1, \ldots, \eta_n$ are complex numbers. Finally, any $h \in \mathcal{H}(B)$ can be uniquely identified by its values at $\mathcal{Z}$,

$$\{h(z_0), h(z_1), \ldots, h(z_n)\}.$$

The correspondence between these three alternative representations is the content of the following lemma.

Lemma 7.1. *Let $G := [g_k(z_j)]_{j,k}$ be the matrix (4.23), and define the Vandermonde matrix $V := [\bar{z}_k^{-j}]_{j,k}$. Then, for any $h \in \mathcal{H}(B)$,*

$$h = V\eta,$$

where $\eta = (\eta_1, \eta_2, \ldots, \eta_n)'$ is defined via (7.5), and

$$\begin{bmatrix} h(z_1) - h_0 \\ h(z_2) - h_0 \\ \vdots \\ h(z_n) - h_0 \end{bmatrix} = G\eta.$$

Moreover, G and V are invertible.

Proof. The first correspondence follows immediately from (7.5) and the expansion

$$g_k(z) = \bar{z}_k^{-1} z^{-1} + \bar{z}_k^{-2} z^{-2} + \bar{z}_k^{-3} z^{-3} + \ldots.$$

The second correspondence also follows from (7.5). Finally, we already established invertibility of G in the proof of Proposition 4.3, and the Vandermonde matrix V is invertible since the points in $\mathcal{Z}$ are distinct. $\square$

We now reformulate the steps of the algorithm given in Section 6 in terms of the real Markov coordinates of the relevant functions. We shall consistently work with functions in $\mathcal{H}(B)$. Therefore, as $f \notin \mathcal{H}(B)$, we form

$$\hat{f} := \Pi_{\mathcal{H}(B)} f,$$

where $\Pi_{\mathcal{H}(B)}$ denotes orthogonal projection onto $\mathcal{H}(B)$. Since $\hat{f} = f + Bg$ for a suitable $g \in \mathcal{H}^2$, it follows that

$$\hat{f}(z_k) = f(z_k) \quad \text{for } k = 0, 1, \ldots, n.$$

Next, define $w(z)$ to be the unique function in $\mathcal{H}(B)$ such that

$$w(z_k) = w_k \quad \text{for } k = 0, 1, \ldots, n.$$

The gradient of φ in Lemma 5.5 can then be expressed in terms of the "error function"

$$r(z) := w(z) - \Pi_{\mathcal{H}(B)} f(z), \tag{7.6}$$

which also belongs to $\mathcal{H}(B)$. In fact,

$$r(z_k) := w_k - f(z_k). \tag{7.7}$$

Moreover, we introduce an $\mathcal{H}(B)$-representation for any $Q \in \mathcal{S}$ and any given $\Psi \in \mathcal{S}$ by writing

$$Q(z) = q(z) + q^*(z), \quad \Psi(z) = \psi(z) + \psi^*(z),$$

where $q, \psi \in \mathcal{H}(B)$ are positive real. Finally, we represent q and ψ by their respective Markov coordinates (x, x_0) and (y, y_0), respectively, in the standard state-space representation described above, i.e.,

$$\mathbf{q} = \left[\begin{array}{c|c} A & x \\ \hline c & x_0 \end{array} \right] \quad \text{and} \quad \psi = \left[\begin{array}{c|c} A & y \\ \hline c & y_0 \end{array} \right].$$

22   C. I. BYRNES, T. T. GEORGIOU, AND A. LINDQUIST

We begin with the state-space implementation of Step 1 in Section 6. In this context, we have the following version of Lemma 5.5.

Proposition 7.2. *Let G and V be defined as in Lemma 7.1. Given an $f \in \mathcal{P}$, let q be the positive real part of $Q := \mathfrak{J}^{-1}f$, where $\mathfrak{J}$ is defined as in (6.4). Moreover, let $r(z)$ be given by (7.6), and denote by (x_0, x) and (r_0, r) the Markov coordinates of $q(z)$ and $r(z)$ respectively. Then*

$$\frac{\partial \varphi}{\partial x_0} = 4r_0,$$

$$\frac{\partial \varphi}{\partial x} = Tr,$$

where $T := (V^*)^{-1}GV^{-1}$ is a real matrix.

Proof. Since $q_0 = 2x_0$ and $w_0 - f(z_0) = r(z_0) = r_0$, the derivative with respect to x_0 follows immediately from (5.2). Next, applying Lemma 7.1, we see that

$$\left[\frac{\partial x_j}{\partial q_k}\right]_{j,k=1}^n = V,$$

and that $r(z_k) - r_0$ is the k:th entry in $GV^{-1}r$. Moreover, by (7.7), we have

$$r(z_k) - r_0 = [w_k - f(z_k)] - [w_0 - f(z_0)]$$

for $k = 1, 2, \ldots, n$. Finally, using equation (5.3) and defining $\bar{q} := (\bar{q}_1, \bar{q}_2, \ldots, \bar{q}_n)'$, we obtain

$$\frac{\partial \varphi}{\partial x} = \left(\left[\frac{\partial x}{\partial \bar{q}}\right]^{-1}\right)' \frac{\partial \varphi}{\partial \bar{q}} = (V^*)^{-1}GV^{-1}r$$

establishing the rest of the proposition. □

It remains to determine the projection $\hat{f} := \Pi_{\mathcal{H}(B)}f$. We present the construction in two steps. Note that, since the points in $\mathcal{Z}$ are assumed to be distinct, $z = 0$ is a simple pole of $B(z)$.

Proposition 7.3. *Assume that $B(z) = z^{-1}B_0(z)$ with $B_0(\infty) \neq 0$, and let $f(z) = f_0 + z^{-1}f_r(z) \in \mathcal{H}_2$, with $f_r(z) \in \mathcal{H}_2$. Then*

$$\Pi_{\mathcal{H}(B)}f = f_0 + z^{-1}\Pi_{\mathcal{H}(B_0)}f_r(z).$$

Proof. First note that, for any $f \in \mathcal{H}_2$,

$$\hat{f} := \Pi_{\mathcal{H}(B)}f = B\Pi_-B^*f,$$

where Π_- is the orthogonal projection onto the $\mathcal{H}_2^\perp$, the orthogonal complement of $\mathcal{H}_2$ in $\mathcal{L}_2$. In fact, $f = \hat{f} + Bq$ for some $q \in \mathcal{H}_2$, and hence $\Pi_-B^*f = B^*\hat{f}$. Then, since $B_0^*zf_0 \in \mathcal{H}_2^\perp$,

$$\hat{f} = z^{-1}B_0\Pi_-B_0^*z(f_0 + z^{-1}f_r)$$

$$= f_0 + z^{-1}B_0\Pi_-B_0^*f_r,$$

from which the proposition immediately follows. □

A GENERALIZED ENTROPY CRITERION FOR RATIONAL INTERPOLATION   23

The second step given below deals with $\Pi_{\mathcal{H}(B_0)} f_r$. Note that, while B is not in $\mathcal{H}(B)$, B_0 is. Therefore, B_0 has a state space representation with A and c given by (7.4). However, $f_r \notin \mathcal{H}(B)$, so we must use other A and c matrices for $\mathbf{f}_r$.

Proposition 7.4. *Assume that $B_0 \in \mathcal{H}_2$ is a rational Blaschke product which is nonzero at infinity, and let*

$$\mathbf{B}_0 = \left[\begin{array}{c|c} A & b \\ \hline c & d \end{array} \right] \quad and \quad \mathbf{f} = \left[\begin{array}{c|c} A_1 & b_1 \\ \hline c_1 & d_1 \end{array} \right].$$

Then the Sylvester equation

$$-(A - bd^{-1}c)X + XA_1 + bd^{-1}c_1 = 0, \tag{7.8}$$

has a unique solution X, and $\hat{f}_r := \Pi_{\mathcal{H}(B_0)} f_r$ has the state-space representation

$$\hat{\mathbf{f}}_r = \left[\begin{array}{c|c} A & b(d_- - d^{-1}d_1) - Xb \\ \hline c & dd_- \end{array} \right], \tag{7.9}$$

where

$$d_- := -d^{-1}c(A - bd^{-1}c)^{-1}(bd^{-1}d_1 + Xb_1). \tag{7.10}$$

Proof. We first note that $B_0^*(z)$ has the state-space description

$$\mathbf{B}_0^* = \left[\begin{array}{c|c} A_0 & b_0 \\ \hline c_0 & d_0 \end{array} \right]$$

with $A_0 = A - bd^{-1}c$, $b_0 = bd^{-1}$, $c_0 = -d^{-1}c$ and $d_0 = d^{-1}$ and with A_0 having all its eigenvalues outside the unit disc. Consider the Lyapunov equation

$$X - A_0^{-1}XA_1 - A_0^{-1}b_0c = 0,$$

which has a unique solution since both A_1 and A_0^{-1} have eigenvalues inside the unit disc [23]. But then so does

$$-A_0X + XA_1 + b_0c_1 = 0,$$

which is the same as (7.8). Using standard manipulations (see, e.g., [22, p.IX]), it follows that $B_0^* f_r$ has the state space form

$$\mathbf{B}_0^* \mathbf{f}_r = \left[\begin{array}{cc|c} A_0 & 0 & b_0d_1 + Xb_1 \\ 0 & A_1 & b_1 \\ \hline c_0 & d_0c_1 - c_0X & d_0d_1 \end{array} \right].$$

Next, consider $\tilde{f}(z) := \Pi_- B_0^* f_r$, which has the representation

$$\tilde{\mathbf{f}} = \left[\begin{array}{c|c} A_0 & b_0d_1 + Xb_1 \\ \hline c_0 & d_- \end{array} \right], \quad d_- := c_0A_0^{-1}(b_0d_1 + Xb_1),$$

where the nontrivial d-term is due to the fact that $\tilde{f}(z)$ being in $\mathcal{H}_2^\perp$ must vanish at the origin. The final step needed to obtain a canonical realization for $\hat{f}_r := B_0\tilde{f}$, is standard and involves cancellation of the unobservable modes at the poles of $\tilde{f}$. Note that these poles coincide with the zeros of B_0. $\square$

24   C. I. BYRNES, T. T. GEORGIOU, AND A. LINDQUIST

The state-space version of Step 2 in Section 6 is developed along the same lines as in Step 1 by instead representing relevant functions in $\mathcal{H}(B^2)$. Then a Newton step is taken as described in Step 3. Alternatively, a gradient method is used, in which case Step 2 can be deleted.

Finally, Step 4, i.e., determining f from q, amounts to solving a matrix Riccati equation and a Lyapunov equation, as seen from the following proposition.

Proposition 7.5. *Suppose that $q, \psi \in \mathcal{H}(B)$ are strictly positive real with Markov parameters (x, x_0) and (y, y_0), respectively. Let P be the unique solution to the algebraic Riccati equation*

$$P = APA' + (x - APc')(2x_0 - cPc')^{-1}(x - APc')', \qquad (7.11)$$

$$d_1 := (2x_0 - cPc')^{1/2},$$

$$b_1 := (x - APc')d_1^{-1},$$

having the property that

$$\Gamma := A - b_1 d_1^{-1} c' \qquad (7.12)$$

is stable, and let X be the unique solution of the Lyapunov equation

$$X = \Gamma X \Gamma' + y y_0^{-1} y' - (y - b_1 d_1^{-1} y_0) y_0^{-1} (y - b_1 d_1^{-1} y_0)', \qquad (7.13)$$

$$d_2 := \tfrac{1}{2}(y_0 - cXc')d_1^{-1},$$

$$b_2 := [(y - Axc') - b_1 d_2]d_1^{-1}.$$

Then $f = \Im(q + q^*)$, defined as in (6.4), has the state-space representation

$$\mathbf{f} = \left[\begin{array}{c|c} \Gamma & b_1 d_1^{-1} d_2 - b_2 \\ \hline -d_1^{-1} c & d_1^{-1} d_2 \end{array} \right].$$

Proof. Observe that determining $a(z)$ from $q + q^* = aa^*$ is a standard spectral factorization problem [2, 20] with the unique minimum-phase solution given by

$$\mathbf{a} = \left[\begin{array}{c|c} A & b_1 \\ \hline c & d_1 \end{array} \right].$$

Then $b(z)$ is determined from the linear equation

$$\Psi = \psi + \psi^* = ab^* + ba^*$$

which, in the state-space formulation, becomes (7.13). Since Γ is stable, it has a unique solution X. Finally, the state space description of $f = a^{-1}b$ is obtained by direct computation, using the formalism in, e.g., [22, p.IX]. $\square$

8. Applications

In this section we describe a number of application for which our theory appears to be especially relevant. We touch upon problems in robust control, in circuit theory and in modeling of stochastic processes. The examples chosen are simple and basic since our aim is only to indicate the spectrum of potential applications of our theory.
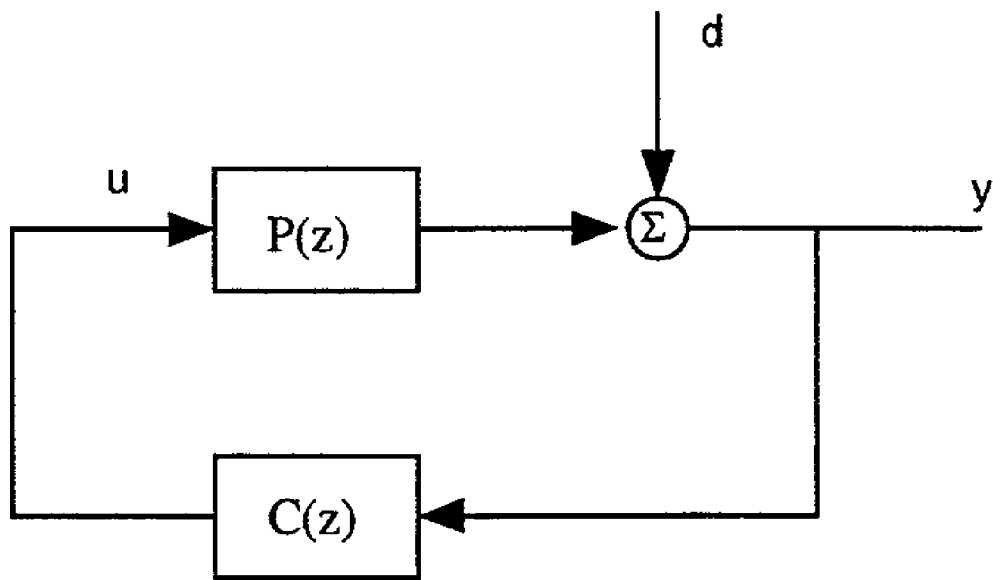
Figure 8.1: Feedback system.

26   C. I. BYRNES, T. T. GEORGIOU, AND A. LINDQUIST

Therefore, for any $\alpha > \alpha_{\mathrm{opt}}$, there are admissible sensitivity functions S such that

$$\tilde{S}(z) = \frac{1}{\alpha} S(z)$$

maps the exterior of the disc into the unit disc. Such a function, which has its zeros outside the unit disc as required, will be called *bounded real.*

In the context of classical control, it is then very convenient to formulate this as a Nevanlinna-Pick interpolation problem, since one could also formulate the Nevanlinna-Pick theory for bounded real functions. In fact, since the linear fractional transformation $s = \frac{1+z}{1-z}$ maps the unit disc into the right half plane, one is lead to finding positive real rational functions

$$f(z) = \frac{\alpha + S(z)}{\alpha - S(z)}$$

of degree at most $n := r + \ell - 1$, which satisfy the interpolation conditions

$$f(p_j) = 1, \quad j = 1, 2, \ldots, \ell \quad \text{and} \quad f(z_i) = \frac{\alpha + 1}{\alpha - 1}, \quad i = 1, 2, \ldots, r.$$

To satisfy additional design specifications, by the Spectral Zero Assignability Theorem (Corollary 4.2), we may then choose the zeros of

$$\alpha^2 - S(z)S^*(z) \tag{8.3}$$

arbitrarily in the open unit disc.

For example, in weighted sensitivity optimization, low sensitivity is required for certain frequencies, and, to account for this in the design, S is replaced by WS for some rational shaping filter $W(z)$. (cf. [22, Chapter 9], [55, Chapter 8]). This results in an increase in the complexity of the problem and an increase in the dimension of the relevant feedback operators by an amount equal to the dimension of the shaping filter. An alternative approach, as suggested above, is to tune the free parameters of our parameterization (i.e., the spectral zeros) to shape the appropriate frequency curve of the system (i.e., loop shape, sensitivity function, etc.). This idea is illustrated in the following example.

Example 1. Consider the discrete-time linear system, shown in Figure 8.1, where $P(z) = \frac{1}{z-2}$, and $C(z)$ is a suitable stabilizing controller such that the sensitivity function (8.1) is first-order and high-pass. The transfer function $P(z)$ has one pole and one zero outside the unit disc, namely a pole at 2 and a zero at ∞. Following notation in previous sections, we set $z_0 = \infty$ and $z_1 = 2$. The sensitivity function $S(z)$ must satisfy the interpolation conditions $S(z_0) = 1$ and $S(z_i) = 0$. It can be shown that $\alpha_{\mathrm{opt}} = 2$, so we take $\alpha = 2.5$.

The standard approach is to choose a weighting filter $W(z)$ describing the inverse of the desired shape for $S(z)$ and considering the new sensitivity function $\hat{S}(z) := W(z)S(z)$. The central solution $\hat{S}(z)$ to the Nevanlinna-Pick problem with new interpolation conditions $\hat{S}(z_0) = W(\infty)$ and $\hat{S}(z_1) = 0$ is then solved, after which $S = W^{-1}\hat{S}$ is reconstructed. While this strategy allows for direct control of the shape of S, it causes its dimension to be enlarged accordingly. In particular, in this example the dimension of S will generically equal the dimension of $W(s)$ plus one. However,
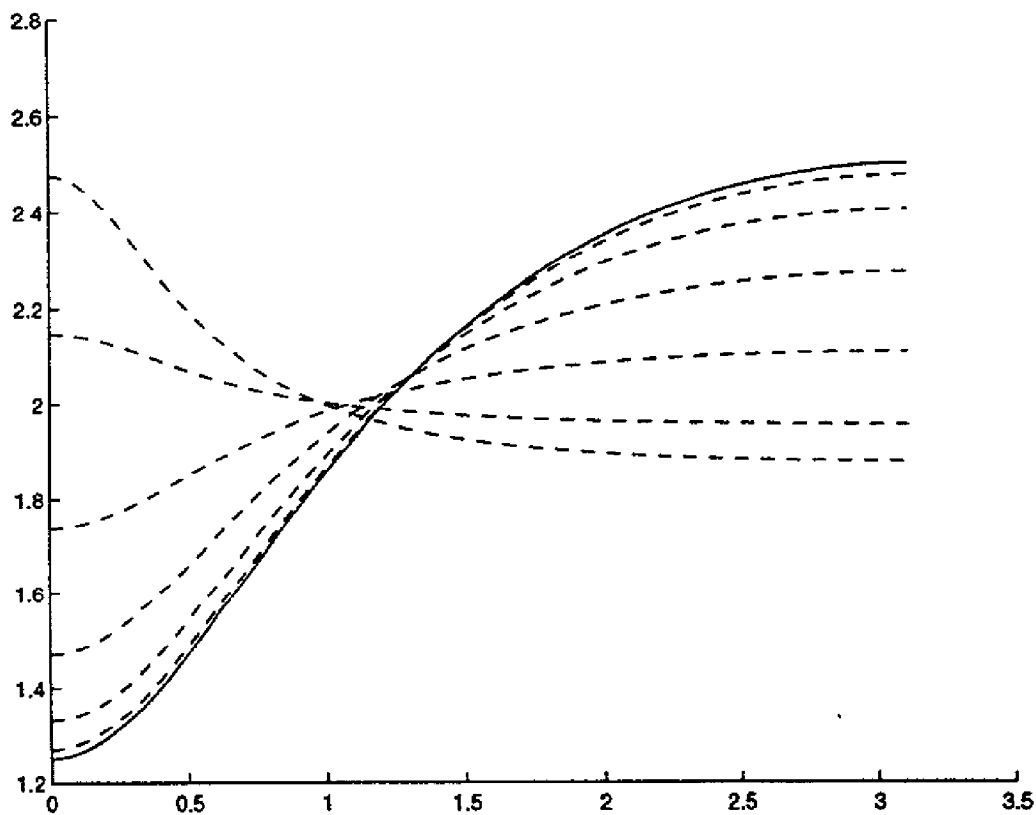
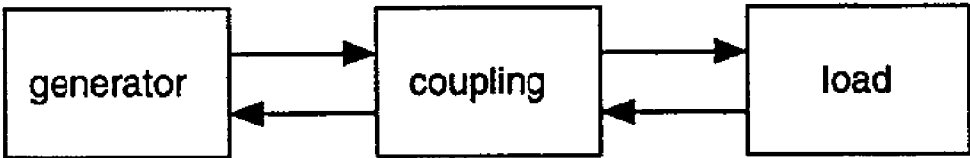Figure 8.2: $|S(e^{i\theta})|$ as a function of θ

Figure 8.3: Two port connection.

Thus, the problem of maximizing the transducer power gain amounts to minimizing the H_∞ norm of $\rho(s)$ subject to the constraints (8.6).

Since the transducer power gain is rarely required to be uniform across frequencies, the usual approach to the problem is to specify a desired transducer power gain shape and then to determine whether a solution is feasible. (See [15, Chapter 4]. Also see Helton [34] for an alternative formulation generalizing Youla's theory.) However, in the context of the theory developed in the present paper, we may instead select the zeros of

$$G(s) := 1 - \rho(s)\rho(-s),$$

i.e., the zeros of the power gain.

Example 2. Consider a passive load with impedance

$$Z_\ell(s) = \frac{1 + R_1 C s}{1 + (1 + R_1)Cs} + \frac{R_2 + Ls}{1 + R_2 + Ls},$$

where $R_1 = 0.5\Omega$, $R_2 = 0.1\Omega$, $L = 0.5H$, and $C = 0.01F$. (This is a cascade connection of two filters, which are the parallel connections of a resistor $R = 1\Omega$ with a lossy capacitor and a lossy inductor respectively.) The transmission zeros of $Z_\ell(s)$ are computed as the zeros of $Z_\ell(s) + Z_\ell(-s)$ to be ± 81.6429, ± 1.6249. The Blaschke factor

$$B(s) = \frac{(1 + R_2 - Ls)(1 - (1 + R_1)Cs)}{(1 + R_2 + Ls)(1 + (1 + R_1)Cs)}$$

evaluated at the transmission zeros provides the interpolation data

$$\rho(81.6429) = 0.0957, \quad \rho(1.6249) = 0.1432.$$

As mentioned in the previous example, the theory of the paper applies to any class of functions which is conformally equivalent to positive real functions. Thus we begin by translating the problem to the "discrete-time setting" via the conformal mapping $s = \frac{z-1}{z+1}$, which maps the right-half-plane bijectively onto the complement of the unit disc. Then a function g in the example corresponds, via the transformation $\hat{g}(z) := g(\frac{1-z}{1+z})$, to a function $\hat{g}$ in the new setting. In this representation, the transducer power gain relates to the magnitude of a suitable bounded real function via

$$\hat{G}(e^{i\theta}) = 1 - |\hat{\rho}(e^{i\theta})|^2.$$

Translating the interpolation data to the z domain we obtain

$$\hat{\rho}(-1.0248) = 0.0957, \quad \hat{\rho}(-4.2003) = 0.1432.$$

Next, the conformal mapping

$$\hat{r}(z) = \frac{1 - \hat{\rho}(z)}{1 + \hat{\rho}(z)}$$

transforms the bounded real function $\hat{\rho}$ to to the positive real function $\hat{r}$. Thus, we seek a positive real function $\hat{r}$ such that

$$\hat{r}(-1.0248) = 1.2116 \quad \hat{r}(-4.2003) = 1.3342.$$

It is important to note that the zeros of $\hat{r} + \hat{r}^*$ are identical to the zeros of $1 - \hat{\rho}\hat{\rho}^*$. These are usually called *spectral zeros*. Finally, it remains to move the point $z_0 = -1.0248$ to ∞. This is done via the linear fractional transformation $z \to \frac{1+1.0248z}{z+1.0248}$, thus defining
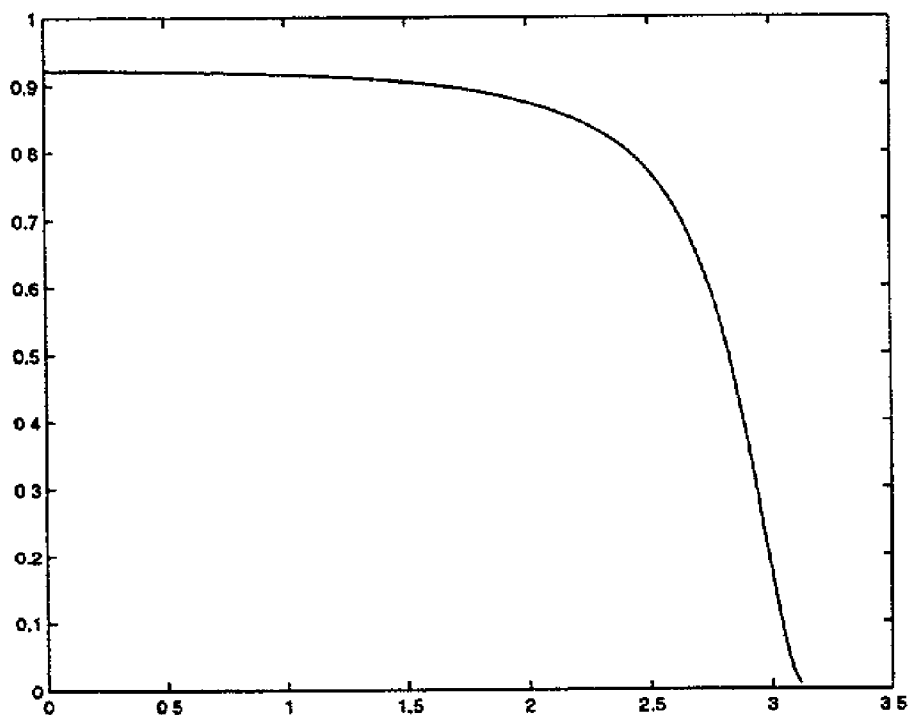
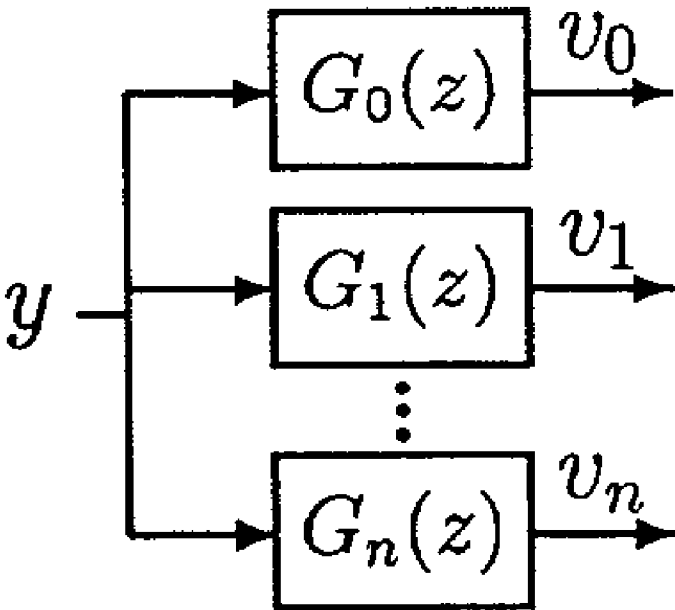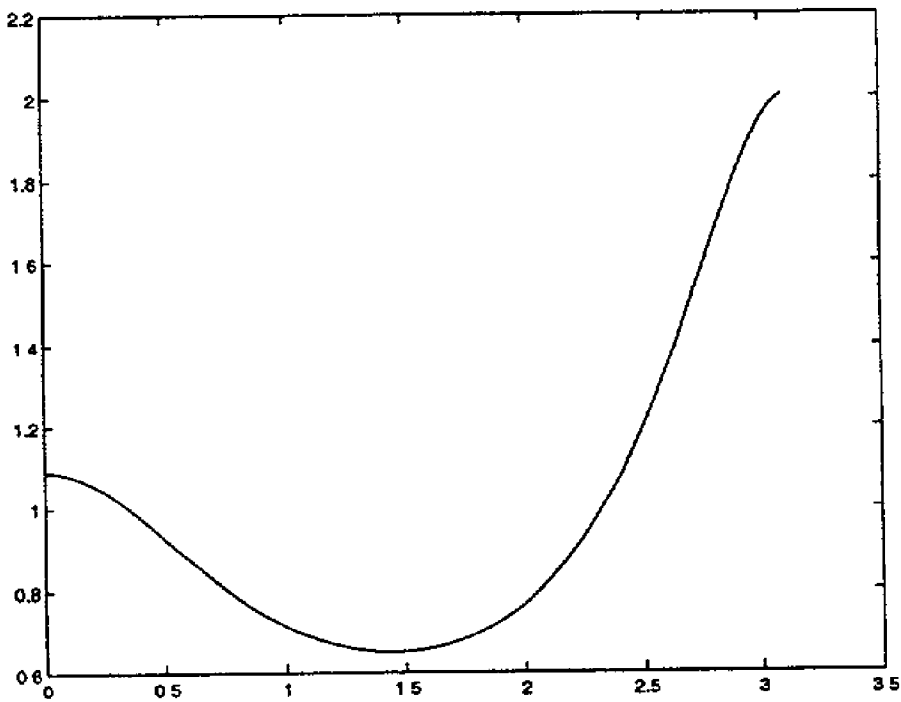Figure 8.4 : $|\hat{G}(e^{i\theta})|$ as a function of θ

Figure 8.5: Filter bank.

Figure 8.6: $|\Phi(e^{i\theta})|$ as a function of θ

defined by the coprime factorizations

$$P_0(z) = \frac{N_0(z)}{D_0(z)} \quad \text{and} \quad P_1(z) = \frac{N_1(z)}{D_1(z)},$$

where $N_k, D_k,\ k = 0, 1$, are stable, proper rational functions. Then the problem at hand is to find a compensator

$$C(z) = \frac{N_c(z)}{D_c(z)}, \tag{8.7}$$

which simultaneously stabilizes not only these two plants but the one-parameter family

$$P_\lambda = \frac{\lambda N_1(z) + (1 - \lambda)N_0(z)}{\lambda D_1(z) + (1 - \lambda)D_0(z)}, \quad \lambda \in [0, 1]$$

of plants. It is easy to see that the compensator (8.7) stabilizes P_λ for all $\lambda \in [0, 1]$ if and only if there are stable, minimum-phase transfer functions Δ_0 and Δ_1 such that

(i) the factors N_c, D_c satisfy the equations

$$D_k(z)D_c(z) - N_k(z)N_c(z) = \Delta_k(z) \quad k = 0, 1 \tag{8.8}$$

(ii) the rational function

$$\lambda \Delta_1 + (1 - \lambda)\Delta_0$$

is stable and minimum-phase for all $\lambda \in [0, 1]$.

Starting with condition (i), we solve the system (8.8) for N_c, D_c to obtain

$$N_c = \frac{\Delta_0 D_1 - \Delta_1 D_0}{N_1 D_0 - N_0 D_1}, \quad D_c = \frac{\Delta_0 N_1 - \Delta_1 N_0}{N_1 D_0 - N_0 D_1}.$$

Suppose that $N_1 D_0 - N_0 D_1$ has zeros at $z_0, z_1, \ldots, z_n$ outside the unit disc, and set

$$w_k := \frac{D_1(z_k)}{D_0(z_k)} = \frac{N_1(z_k)}{N_0(z_k)}, \quad k = 0, 1, \ldots, n. \tag{8.9}$$

Then, for D_c and N_c to be stable as required, $z_0, z_1, \ldots, z_n$ must also be zeros of $\Delta_0 D_1 - \Delta_1 D_0$ and $\Delta_0 N_1 - \Delta_1 N_0$, which happens if and only if

$$\frac{\Delta_1(z_k)}{\Delta_0(z_k)} = w_k, \quad k = 0, 1, \ldots, n \tag{8.10}$$

Next, condition (ii) requires that $\lambda \Delta_1(z) + (1 - \lambda)\Delta_0(z) \neq 0$ for all $z \in \mathbb{D}^c$, or equivalently

$$\frac{\Delta_1(z)}{\Delta_0(z)} \neq \frac{\lambda - 1}{\lambda}, \quad \lambda \in [0, 1], \quad z \in \mathbb{D}^c, \tag{8.11}$$

which excludes the whole negative real line.

Consequently, the problem is reduced to finding a rational function

$$F(z) := \frac{\Delta_1(z)}{\Delta_0(z)}$$

mapping $\mathbb{D}^c$ into the complex plane minus the negative real line and satisfying the interpolation conditions (8.10). Then

$$C(z) = \frac{D_1(z) - F(z)D_0(z)}{N_1(z) - F(z)N_0(z)}$$

is an admissible compensator. However, $f(z) := (F(z))^{1/2}$ maps $\mathbb{D}^c$ into the open right half plane and is therefore positive real. Hence, we may instead apply the methods of this paper to determine a positive real function f of degree at most n satisfying the interpolation conditions

$$f(z_k) = (w_k)^{1/2}, \quad k = 0, 1, \ldots, n,$$

and then take $F(z) = f(z)^2$. The different solutions are parameterizes by the spectral zeros of f, i.e., the zeros of $f + f^*$.

9. Conclusions

In this paper, we have given a method for finding all solutions to the scalar, rational Nevanlinna-Pick interpolation problem, having degree less than or equal to n, in terms of the minima of a parameterized family of convex optimization problems. While the problem has been posed for positive real interpolants, as would arise for the control of discrete-time systems, standard linear fractional transformations can adapt this generalized entropy criterion approach to positive real, or bounded-real, transfer functions for both continuous and discrete-time linear systems.

Appendix A. Proofs of Propositions 5.1, 5.2 and 5.4

Proof of Proposition 5.1. We want to prove that $\varphi(q)$ is finite when $q \neq 0$. Then the rest follows by inspection. Clearly, $\varphi(q)$ cannot take the value $-\infty$; hence it remains to prove that $\varphi(q) < \infty$. Since $q \neq 0$,

$$\mu := \max_\theta Q(e^{i\theta}) > 0.$$

Then, setting $P(z) := \mu^{-1}Q(z)$,

$$\log P(e^{i\theta}) \leq 0 \tag{A.1}$$

and

$$\varphi(q) = \varphi_1(q) - \frac{1}{2\pi}\log\mu \int_{-\pi}^{\pi} \Psi(e^{i\theta})d\theta - \frac{1}{2\pi}\int_{-\pi}^{\pi} \log[P(e^{i\theta})]\Psi(e^{i\theta})d\theta,$$

and hence the question of whether $\varphi(q) < \infty$ is reduced to determining whether

$$-\int_{-\pi}^{\pi} \log[P(e^{i\theta})]\Psi(e^{i\theta})d\theta < \infty.$$

But, since $\Psi(e^{i\theta}) \leq M$ for some bound M, this follows from

$$\int_{-\pi}^{\pi} \log P(e^{i\theta})d\theta > -\infty, \tag{A.2}$$

which is the well-known Szegö condition: (A.2) is a necessary and sufficient condition for $P(e^{i\theta})$ to have a stable spectral factor [29]. But, since the rational function $P(z)$ belongs to $\mathbb{S}$, satisfies $P(z) = P^*(z)$ and is nonnegative on the unit circle, there is a

A GENERALIZED ENTROPY CRITERION FOR RATIONAL INTERPOLATION   35

function $\pi(z) \in \mathcal{H}(B)$ such that $\pi(z)\pi^*(z) = P(z)$. But then $\pi(z)$ is a stable spectral factor of $P(z)$, and hence (A.2) holds. $\square$

Proof of Proposition 5.2. Suppose $q^{(k)}$ is a sequence in $M_r := \varphi^{-1}(-\infty, r]$. It suffices to show that $q^{(k)}$ has a convergent subsequence. The sequence $q^{(k)}$ defines a sequence of unordered n-tuples of zeros lying in the the unit disc, and a sequence of scalar multipliers. We wish to prove that both of these sequences cluster. To this end, each $Q^{(k)}$ may be factored as

$$Q^{(k)}(z) = \lambda_k a_k(z)a_k^*(z) = \lambda_k \tilde{Q}^{(k)}(z),$$

where λ_k is positive and $a_k(z)$ is a function in $\mathcal{H}(B)$ which is normalized so that $a_k(\infty) = 1$.

We shall first show that the sequence of zeros clusters. The corresponding sequence of the (unordered) set of n zeros of each $a_k(z)$ has a convergent subsequence, since all (unordered) sets of zeros lie in the closed unit disc. Denote by $a(z)$ the function in $H(B)$ which vanishes at this limit set of zeros and which is normalized so that $a(\infty) = 1$. By reordering the sequence if necessary, we may assume the sequence $a_k(z)$ tends to $a(z)$. Therefore the sequence $q^{(k)}$ has a convergent subsequence if and only if the sequence λ_k does.

We now show that the sequence of multipliers, λ_k, clusters. It suffices to prove that the sequence λ_k is bounded from above and from below away from zero. This will follow by analyzing the linear and the logarithmic growth in

$$\varphi(q^{(k)}) = \varphi_1(\tilde{q}^{(k)}) - \frac{1}{2\pi} \log \lambda_k \int_{-\pi}^{\pi} \Psi(e^{i\theta})d\theta - \frac{1}{2\pi} \int_{-\pi}^{\pi} \log[\tilde{Q}^{(k)}(e^{i\theta})]\Psi(e^{i\theta})d\theta$$

with respect to the sequence λ_k. Here $\varphi_1(q)$ is the linear term (5.1) of $\varphi(q)$. We first note that the sequence $\varphi_1(\tilde{q}^{(k)})$, where $\tilde{q}^{(k)}$ is the vector corresponding to the pseudo-polynomial $\tilde{Q}^{(k)}$, is bounded from above because the normalized functions $a_k(z)$ lie in a bounded set. Similarly, by the proof of Lemma 5.3, the sequence $\varphi_1(\tilde{q}^{(k)})$ is bounded from below, away from zero. In particular, the coefficient of λ_k in the first term for this expression for $\varphi(q^{(k)})$ is bounded away from 0 and away from ∞. We also note that the coefficient of $\log \lambda_k$ in this expression for $\varphi(q^{(k)})$ is independent of k. Next, the term

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log[\tilde{Q}^{(k)}(e^{i\theta})]\Psi(e^{i\theta})d\theta \qquad (A.3)$$

in this expression for $\varphi(q^{(k)})$ is independent of λ_k, and we claim that it remains bounded as a function of k. Indeed, are both bounded from above and from below respectively away from zero and $-\infty$. The upper bounds come from the fact that $\text{Re}\langle w+1, \tilde{q}^{(k)}\rangle$ are Schur polynomials and hence have their coefficients in the bounded Schur region. In fact,

$$\tilde{Q}^{(k)}(e^{i\theta}) \rightarrow |a(e^{i\theta})|^2 = Q(z)$$

where $a(z)$ has all its zeros in the closed unit disc. In particular, if q in $\overline{\mathcal{Q}}_n$ corresponds to q, then the third term in the expression for $\varphi(q^{(k)})$ converges to $\varphi(q)$, which is finite since a is not identically zero.

36   C. I. BYRNES, T. T. GEORGIOU, AND A. LINDQUIST

Finally, observe that if a subsequence of λ_k were to tend to zero, then $\varphi(q^{(k)})$ would exceed r. Likewise, if a subsequence of λ_k were to tend to infinity, φ would exceed r, since linear growth dominates logarithmic growth. $\square$

Proof of Proposition 5.4. Denoting by $D_p\varphi(q)$ the directional derivative of φ at q in the direction p, it is easy to see that

$$D_p\varphi(q) := \lim_{\epsilon \to 0} \frac{\varphi(q + \epsilon p) - \varphi(q)}{\epsilon}$$

$$= \varphi_1(p) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{P(e^{i\theta})}{Q(e^{i\theta})} \Psi(e^{i\theta}) d\theta, \qquad (A.4)$$

where $P(z)$ is the pseudo-polynomial

$$P(z) = \bar{p}_n g_n^*(z) + \ldots + \bar{p}_1 g_1^*(z) + p_0 g_0(z) + p_1 g_1(z) + \ldots + p_n g_n(z)$$

corresponding to the vector $p \in \mathbb{C}^{n+1}$. In fact,

$$\frac{\log(Q + \epsilon P) - \log Q}{\epsilon} = \frac{P}{Q} \log \left[(1 + \epsilon \frac{P}{Q})^{\frac{1}{\epsilon} \frac{Q}{P}} \right] \to \frac{P}{Q}$$

as $\epsilon \to +0$, and hence (A.4) follows by dominated convergence.

Now, let $q \in \mathfrak{Q}$ and $\tilde{q} \in \partial\mathfrak{Q}$ be arbitrary. Then the corresponding pseudo-polynomials Q and $\tilde{Q}$ have the properties

$$Q(e^{i\theta}) > 0 \quad \text{for all } \theta \in [-\pi, \pi]$$

and

$$\tilde{Q}(e^{i\theta}) \geq 0 \quad \text{for all } \theta \text{ and } \tilde{Q}(e^{i\theta_0}) = 0 \text{ for some } \theta_0.$$

Since $q_\lambda := \tilde{q} + \lambda(q - \tilde{q}) \in \mathfrak{Q}$ for $\lambda \in (0, 1]$, we also have for $\lambda \in (0, 1]$ that

$$Q_\lambda(e^{i\theta}) := \tilde{Q}(e^{i\theta}) + \lambda[Q(e^{i\theta}) - \tilde{Q}(e^{i\theta})] > 0, \quad \text{for all } \theta \in [-\pi, \pi],$$

and we may form the directional derivative

$$D_{\tilde{q}-q}\varphi(q_\lambda) = \varphi_1(\tilde{q} - q) + \frac{1}{2\pi} \int_{-\pi}^{\pi} h_\lambda(\theta) d\theta, \qquad (A.5)$$

where

$$h_\lambda(\theta) = -\frac{Q(e^{i\theta}) - \tilde{Q}(e^{i\theta})}{Q_\lambda(e^{i\theta})} \Psi(e^{i\theta}).$$

Now,

$$\frac{d}{d\lambda} h_\lambda(\theta) = \frac{[Q(e^{i\theta}) - \tilde{Q}(e^{i\theta})]^2}{Q_\lambda(e^{i\theta})^2} \Psi(e^{i\theta}) \geq 0,$$

and hence $h_\lambda(\theta)$ is a monotonely nondecreasing function of λ for all $\theta \in [-\pi, \pi]$. Consequently h_λ tends pointwise to h_0 as $\lambda \to 0$. Therefore,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} h_\lambda(\theta) d\theta \to +\infty \quad \text{as } \lambda \to 0. \qquad (A.6)$$

In fact, if

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} h_\lambda(\theta) d\theta \to \alpha < \infty \quad \text{as } \lambda \to 0, \qquad (A.7)$$

A GENERALIZED ENTROPY CRITERION FOR RATIONAL INTERPOLATION   37

then $\{h_\lambda\}$ is a Cauchy sequence in $L^1(-\pi, \pi)$ and hence has a limit in $L^1(-\pi, \pi)$ which must equal h_0 a.e. But h_0, having poles in $[-\pi, \pi]$, is not summable and hence, as claimed, (A.7) cannot hold.

Consequently, by virtue of (A.5),

$$D_{q-\tilde{q}}\varphi(q_\lambda) \to +\infty \quad \text{as } \lambda \to 0$$

for all $q \in \Omega$ and $\tilde{q} \in \partial\Omega$, and hence, in view of Lemma 26.2 in [44], φ is essentially smooth. Then it follows from Theorem 26.3 in [44] that the subdifferential of φ is empty on the boundary of Ω, and therefore φ cannot have a minimum there. $\square$

References

1. N. I. Akhiezer, *The Classical Moment Problem*, Hafner, 1965.
2. B. D. O. Anderson, *The inverse problem of stationary convariance generation*, J. Statistical Physics **1** (1969), 133–147.
3. J. A. Ball, I. Gohberg and L. Rodman, *Interpolation of Rational Matrix Function*, Birhäuser Verlag, Basel, 1990.
4. V. Belevitch, *Classical Network Theory*, Holden-Day, 1968.
5. J. P. Burg, *A new analysis technique for time series data*, NATO Advanced Institute on Signal Processing with Emphasis on Underwater Acoustics, 1968, reprinted in **Modern Spectrum Analysis**, D.G. Childers, ed., IEEE Press.
6. C. I. Byrnes, T. T. Georgiou, and A. Lindquist, *Maximum entropy spectral estimation using filter banks*, in preparation.
7. C. I. Byrnes and A. Lindquist, *On duality between filtering and control*, in *Systems and Control in the Twenty-First Century*, C. I. Byrnes, B. N. Datta, D. S. Gilliam and C. F. Martin, editors, Birhäuser, 1997, pp. 101–136.
8. C. I. Byrnes and A. Lindquist, *On the partial stochastic realization problem*, IEEE Trans. Automatic Control AC-42 (1997), 1049–1069.
9. C. I. Byrnes, A. Lindquist, S. V. Gusev, and A. S. Mateev, *A complete parametrization of all positive rational extensions of a covariance sequence*, IEEE Trans. Automatic Control AC-40 (1995), 1841–1857.
10. C. I. Byrnes, S. V. Gusev and A. Lindquist, *A convex optimization approach to the rational covariance extension problem*, SIAM J. Control and Optimization, to be published.
11. C. I. Byrnes, H. J. Landau and A. Lindquist, *On the well-posedness of the rational covariance extension problem*, in *Current and Future Directions in Applied Mathematics*, M. Alber, and B. Hu, J. Rosenthal, editors, Birhäuser Boston, 1997, pp. 83–108.
12. C. I. Byrnes, A. Lindquist and Y. Zhou, *On the nonlinear dynamics of fast filtering algorithms*, SIAM J. Control and Optimization, 32(1994), 744–789.
13. C. Carathéodory, *Über den Variabilitätsbereich der Koeffizienten von Potenzreihen, die gegebene Werte nicht annehmen*, Math. Ann. **64** (1907), 95–115.
14. C. Carathéodory, *Über den Variabilitätsbereich der Fourierschen Konstanten von positiven harmonischen Functionen*, Rend. di Palermo **32** (1911), 193–217.
15. W-K. Chen, *Theory and Design of Broadband Matching Networks*, Pergamon Press, 1976.
16. Ph. Delsarte, Y. Genin and Y. Kamp, *On the role of the Nevanlinna-Pick problem in circuits and system theory*, Circuit Theory and Applications **9** (1981), 177–187.
17. Ph. Delsarte, Y. Genin, Y. Kamp and P. van Dooren, *Speech modelling and the trigonometric moment problem*, Philips J. Res. **37** (1982), 277–292.
18. J. C. Doyle, B. A. Frances and A. R. Tannenbaum, *Feedback Control Theory*, Macmillan Publ. Co., New York, 1992.
19. P. Dewilde and H. Dym, ?????????????
20. P. Faurre, M. Clerget, and F. Germain, *Opérateurs Rationnels Positifs*, Dunod, 1979.

38   C. I. BYRNES, T. T. GEORGIOU, AND A. LINDQUIST

21. C. Foias and A. E. Frazho, *The Commutant Lifting Approach to Interpolation Problems*, Birhäuser Verlag, Basel, 1990.

22. B. A. Francis, *A Course in H_∞ Control Theory*, Springer-Verlag, 1987.

23. F. R. Gantmacher, *The Theory of Matrices*, Chelsea, New York, 1959.

24. T. T. Georgiou, *Partial realization of covariance sequences*, CMST, Univ. Florida, Gainesville, 1983.

25. T. T. Georgiou, *Realization of power spectra from partial covariance sequences*, IEEE Transactions Acoustics, Speech and Signal Processing **ASSP-35** (1987), 438–449.

26. T. T. Georgiou, *A topological approach to Nevanlinna-Pick interpolation*, SIAM J. Math. Analysis **18** (1987), 1248–1260.

27. T. T. Georgiou, *The interpolation problem with a degree constraint*, preprint.

28. B. K. Ghosh, *Transcendental and interpolation methods in simultaneous stabilization and simultaneous partial pole placement problems*, SIAM J. Control and Optim. **24** (1986), 1091–1109.

29. U. Grenander and M. Rosenblatt, *Statistical Analysis of Stationary Time Series*, Almqvist & Wiksell, Stockholm, 1956.

30. U. Grenander and G. Szegö, *Toeplitz forms and their applications*, Univ. California Press, 1958.

31. S. Haykin, *Nonlinear Methods of Spectral Analysis*, Springer-Verlag, 1983.

32. H. Helson, *Lectures on Invariant Subspaces*, Academic Press, New York, 1964.

33. J. W. Helton, *Non-Euclidean analysis and electronics*, Bull. Amer. Math. Soc. **7** (1982), 1–64.

34. J. W. Helton, *The distance from a function to H^∞ in the Poincaré metric: Electrical power transfer*, J. of Functional Analysis, **38** (1980), 273–314.

35. R. E. Kalman, *Realization of covariance sequences*, Proc. Toeplitz Memorial Conference (1981), Tel Aviv, Israel, 1981.

36. S. A. Kassam and H. V. Poor, *Robust techniques for signal processing*, Proceedings IEEE **73** (1985), 433–481.

37. P. P. Khargonekar and A. Tannenbaum, *Non-Euclidean metrics and robust stabilization of systems with parameter uncertainty*, IEEE Trans. Automatic Control **AC–30** (1985), 1005–1013.

38. H. Kimura, *Positive partial realization of covariance sequences*, Modelling, Identification and Robust Control (C. I. Byrnes and A. Lindquist, eds.), North-Holland, 1987, pp. 499–513.

39. H. Kimura, *Robust stabilizability for a class of transfer functions* IEEE Trans. Automatic Control **AC–29** (1984), 788–793.

40. D. G. Luenberger, *Linear and Nonlinear Programming* (Second Edition), Addison-Wesley Publishing Company, Reading, Mass., 1984.

41. M. Minoux, Jr., *Mathematical Programming: Theory and Algorithms*, Wiley, 1986.

42. D. Mustafa and K. Glover, *Minimum Entropy H^∞ Control*, Springer Verlag, Berlin, 1990.

43. B. Porat, *Digital Processing of Random Signals*, Prentice Hall, 1994.

44. R. T. Rockafellar, *Convex Analysis*, Princeton University Press, 1970.

45. L.R. Rabiner and R.W. Schafer, *Digital Processing of Speech Signals*, Prentice Hall, Englewood Cliffs, N.J., 1978.

46. D. Sarason, *generalized interpolation in H^∞*, Trans. Amer. Math. Soc. **127** (1967), 179–203.

47. I. Schur, *On power series which are bounded in the interior of the unit circle I and II*, Journal fur die reine und angewandte Mathematik **148** (1918), 122–145.

48. A. Tannenbaum, *Feedback stabilization of linear dynamical plants with uncertainty in the gain factor*, Int. J. Control **32** (1980), 1–16.

49. A. Tannenbaum, *Modified Nevanlinna-Pick interpolation of linear plants with uncertainty in the gain factor*, Int. J. Control **36** (1982), 331–336.

50. O. Toeplitz, *Über die Fouriersche Entwicklung positiver Funktionen*, Rendiconti del Circolo Matematico di Palermo **32** (1911), 191–192.

51. J. L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, Amer. Math.Soc. Colloquium Publications, **20**, Providence, R. I., 1956.

52. D. C. Youla and M. Saito, *Interpolation with positive-real functions*, J. Franklin Institute **284** (1967), 77–108.

53. G. Zames, *Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms, and approximate inverses*, IEEE Trans. on Aut. Control, **AC-26**(1981), 301-320 .

A GENERALIZED ENTROPY CRITERION FOR RATIONAL INTERPOLATION 39

54. G. Zames and B. A. Francis, *Feedback, minimax sensitvity, and optimal robust robustness*, IEEE Transactions on Automatic Control **AC–28** (1983), 585–601.

55. K. Zhou, *Essentials of Robust Control*, Prentice-Hall, 1998.

APPENDIX E

110

A NEW APPROACH TO SPECTRAL ESTIMATION: A TUNABLE HIGH-RESOLUTION SPECTRAL ESTIMATOR*

CHRISTOPHER I. BYRNES[†], TRYPHON T. GEORGIOU[§], AND ANDERS LINDQUIST[‡]

August 17, 1998

ABSTRACT. Traditional maximum entropy spectral estimation determines a power spectrum from covariance estimates. The approach presented here is based on the use of general filter banks as a means of obtaining spectral interpolation data. Such data encompass standard covariance estimates. A constructive approach for obtaining suitable pole-zero (ARMA) models from such data is presented. The choice of the zeros (MA-part) of the model is completely arbitrary. By suitably choices of filter-bank poles and spectral zeros the estimator can be tuned to exhibit high resolution in targeted regions of the spectrum.

1. Introduction

In this paper we present a novel approach to spectral estimation. The paper is a companion publication to [11] where the mathematical aspects of our theory have been worked out, and is the culmination of efforts by the authors over a number of years [3]–[11], and [17]–[20].

The approach leads to a Tunable High REsolution Estimator (THREE), based on three elements, namely (i) a bank of filters, (ii) a theory for parameterizing the complete set of spectra which are consistent with the "filter measurements" and have bounded complexity, and (iii) a convex-optimization approach for constructing spectra described in (ii).

The bank of filters is used to process, in parallel, the observation record and obtain estimates of the power spectrum at desired points. These points relate to the filter-bank poles and can be selected to give increased resolution over desired frequency bands. The theory in (ii) implies that a second set of tunable parameters are given by so-called spectral zeros which determine the Moving-Average (MA) parts of the solutions. The solutions turn out to be spectra of Auto-Regressive/Moving-Average (ARMA) filters of complexity at most equal to the dimension of the filter bank.

2   C. I. BYRNES, T. T. GEORGIOU, AND A. LINDQUIST

THREE is especially suitable for being applied to short observation records. We demonstrate the applicability of the approach by several case studies, including identifying spectral lines and estimating power spectra with steep variations.

The structure of the paper is as follows. In Section 2 we introduce the bank of filters, and discuss how the covariances of their outputs provide estimates of the power spectrum at the reflected pole positions. The variability of such statistical estimates and how they are affected by the position of the poles is briefly considered. The section concludes with a motivating example which present a simulation study comparing THREE with traditional AR filtering and with periodogram analysis. Section 3 presents the basic elements of analytic interpolation that are relevant to the current problem. The classical results are reviewed first, and then our recent theory of analytic interpolation with degree constraint is explained. This is continued in Section 4, where the convex optimization approach is presented. This is based on a generalized concept of entropy and leads to state-space formulae for the bounded-complexity interpolants. We conclude the section with a derivation of the special (linear) case of the central interpolant of the classical theory. Section 5 contains several case studies. Certain mathematical facts are discussed in the appendix.

2. Framework for spectral estimation

Let
$$\{y(t) \mid t = \ldots, -1, 0, 1, \ldots\}$$
denote a real-valued, zero-mean, stationary (Gaussian) stochastic process with power spectral density $\Phi_y(e^{i\theta})$ for $\theta \in [-\pi, \pi]$. Throughout this work we assume that y is a scalar process. The vector-case will be the subject of a future study. We shall consider the basic problem of estimating the spectrum Φ_y based on finite observation records

$$\{y_0, y_1, y_2, \ldots, y_N\} \tag{2.1}$$

of the process.

Traditional modern spectral estimation techniques rely on estimates of a number of covariance lags

$$c_0, c_1, c_2, \ldots, c_n, \qquad \text{where } c_k := \mathrm{E}\{y(t)y(t+k)\}, \tag{2.2}$$

and $\mathrm{E}\{\cdot\}$ denotes mathematical expectation. Typically, these are estimated either by suitable averaging over time of the products $y_t y_{t+k}$, or through estimating the partial autocorrelation coefficients first, by different averaging schemes such as Burg algorithm. In either case, the statistical reliability of such estimates decreases for higher order lags due to the fact that averaging takes place over a shorter list of such cross-products.

The primal object of this paper is the function

$$f_y(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_y(e^{i\theta}) \frac{z + e^{-i\theta}}{z - e^{-i\theta}} d\theta, \tag{2.3}$$

about which much is known. It is analytic in $|z| > 1$ and has positive real part there – such functions are called *positive-real*. In fact, the spectral density is

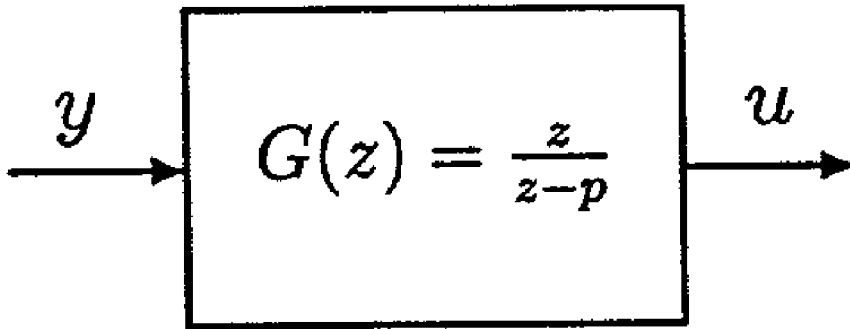$$\Phi_y(e^{i\theta}) = \mathrm{Re}\{f_y(e^{i\theta})\}, \tag{2.4}$$
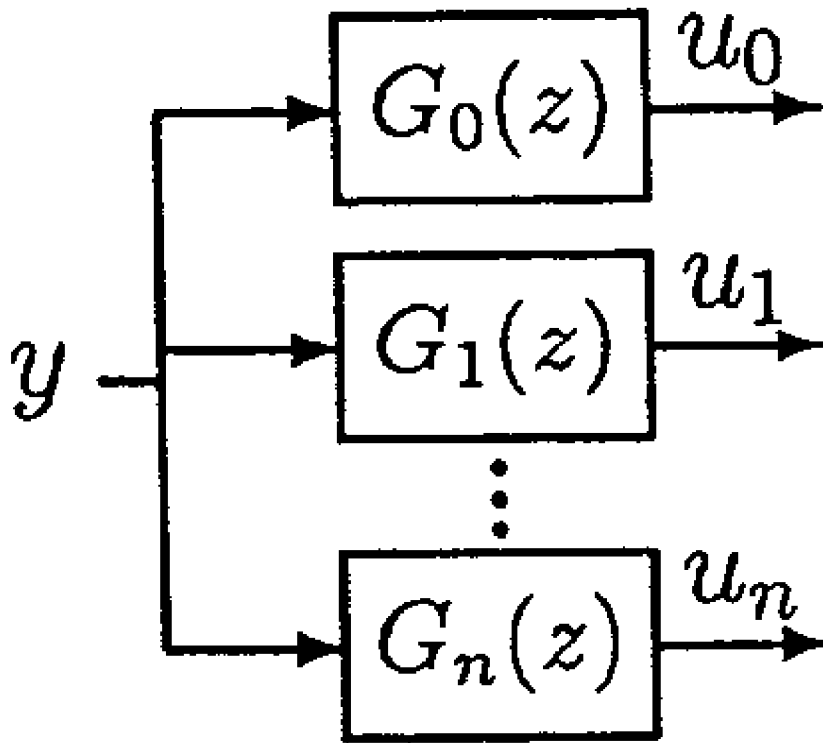
Figure 1: First-order filter.

Figure 2: Bank of filters.

A NEW APPROACH TO SPECTRAL ESTIMATION   5

2.4. The Pick matrix. A central object in analytic interpolation theory is the so-called *Pick matrix*. This matrix arises naturally in the context of our filter bank as the covariance of the output vector-valued process

$$\mathbf{u}(t) := \begin{bmatrix} u_0(t) \\ u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix}.$$

In fact,

$$P_n = \mathrm{E}\{\mathbf{u}(t)\bar{\mathbf{u}}(t)'\} = \begin{bmatrix} \frac{w_0+\bar{w}_0}{1-p_0\bar{p}_0} & \frac{w_0+\bar{w}_1}{1-p_0\bar{p}_1} & \cdots & \frac{w_0+\bar{w}_n}{1-p_0\bar{p}_n} \\ \frac{w_1+\bar{w}_0}{1-p_1\bar{p}_0} & \frac{w_0+\bar{w}_1}{1-p_n\bar{p}_1} & \cdots & \frac{w_0+\bar{w}_n}{1-p_0\bar{p}_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{w_n+\bar{w}_0}{1-p_n\bar{p}_0} & \frac{w_n+\bar{w}_1}{1-p_n\bar{p}_1} & \cdots & \frac{w_n+\bar{w}_n}{1-p_0\bar{p}_n} \end{bmatrix}, \qquad (2.11)$$

where, as usual,

$$w_k = f_y(p_k^{-1}) \quad k = 0, 1, \ldots n.$$

Thus, an alternative way to estimate $f_y(p_k^{-1})$ is through estimates of P_n as a sample covariance of $\mathbf{u}(t)$.

In this paper we only consider distinct points $p_0, p_1, \ldots, p_n$. The general case will be presented elsewhere [12]. For example, the usual Toeplitz matrix

$$T_n = \begin{bmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_0 & \cdots & c_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n-1} & \cdots & c_0 \end{bmatrix}, \qquad (2.12)$$

formed from the partial covariance sequence (2.2) is the Pick matrix for the case in which $p_0 = p_1 = \cdots = p_n = 0$, and hence the filters in our bank are chosen as

$$G_k(z) = z^{-k} \quad k = 0, 1, \ldots, n. \qquad (2.13)$$

This is the case considered in usual AR modeling from covariance data.

2.5. Statistical considerations. This brings us to the statistical reasons for our new approach. In fact, for AR modeling from covariance estimates we need to estimate the Toeplitz matrix (2.12) from the data record (2.1). If this is done via

$$\hat{T}_n = \frac{1}{N-n} \sum_{t=n}^{N} \begin{pmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-n} \end{pmatrix} (y_t, \ y_{t-1}, \ \ldots, \ y_{t-n}),$$

where $\hat{\ }$ denotes "the sample estimate of", then a significant portion of the data has not been fully utilized in estimating lower order covariances due to the large time-lag of some of the filters. Moreover, $\hat{T}_n$ is not in general a Toeplitz matrix. If, instead, we use the covariance estimate

$$\hat{c}_k = \frac{1}{N-k} \sum_{t=k}^{N} y_t y_{t-k},$$

the corresponding Toeplitz matrix may not be positive definite, something that may be rectified by dividing by $N + 1$ rather than $N - k$, by windowing or, by using Burg's algorithm. In any case, any of these methods suffers from the drawback that reliability of the estimate $\hat{c}_k$ of the covariance lag c_k decreases considerably as k grows, especially for relatively short time series [27].

By way of contrast, our method requires only estimating the zeroth covariance lag, or possibly the first covariance lag, in the complex case. It is known the sample variance of covariance estimate

$$\hat{c}_0 := \frac{1}{N+1} \sum_{t=0}^{N} y_t^2$$

is given by

$$\mathrm{var}(\hat{c}_0) = \frac{2}{N+1} \sum_{k=-\infty}^{\infty} c_k^2.$$

(See [27, Section 48.1, Equation (48.6)].) But, using Parseval's theorem, this can be expressed in terms of the spectral density $\Phi_y(e^\theta)$ as follows:

$$\mathrm{var}(\hat{c}_0) = \frac{1}{\pi(N+1)} \int_{-\pi}^{\pi} |\Phi_y(e^\theta)|^2 d\theta.$$

Therefore, for the moment ignoring any transient effects and assuming that the output process u of a filter $G(z)$ driven by y is stationary, the sample variance of the estimate

$$\hat{c}_0(u) := \frac{1}{N+1} \sum_{t=0}^{N} u_t^2 \tag{2.14}$$

becomes

$$\mathrm{var}(\hat{c}_0(u)) = \frac{1}{\pi(N+1)} \int_{-\pi}^{\pi} |G(e^{i\theta})|^2 |\Phi_y(e^\theta)|^2 d\theta. \tag{2.15}$$

This quantifies the effect of the frequency response of $G(z) = \frac{z}{z-p}$ on the variance of statistical estimators for $f_u(p^{-1})$ when estimated by (2.14). In the simple case where $\Phi_y(e^\theta)$ is constant,

$$\mathrm{var}(\hat{c}_0(u)) = \frac{2}{N+1} \frac{1}{1 - |p|^2}.$$

In general, the shape of $|G(e^{i\theta})|$ and its relation to $|\Phi_y(e^\theta)|$ has a direct effect on $\mathrm{var}(\hat{c}_0(u))$ as shown in (2.15). The general conclusion, however, is that choosing the filter poles too close to the unit circle may produce larger errors. Such a strategy will also produce more accentuated transients and is therefore not without cost.

2.6. A motivating example. We conclude this section with a motivating example, which shows that an appropriate choice of filter poles, will considerably increase the resolution of the estimated spectral density for a targeted part of the spectrum when identifying periodic components of a signal from short observation records.
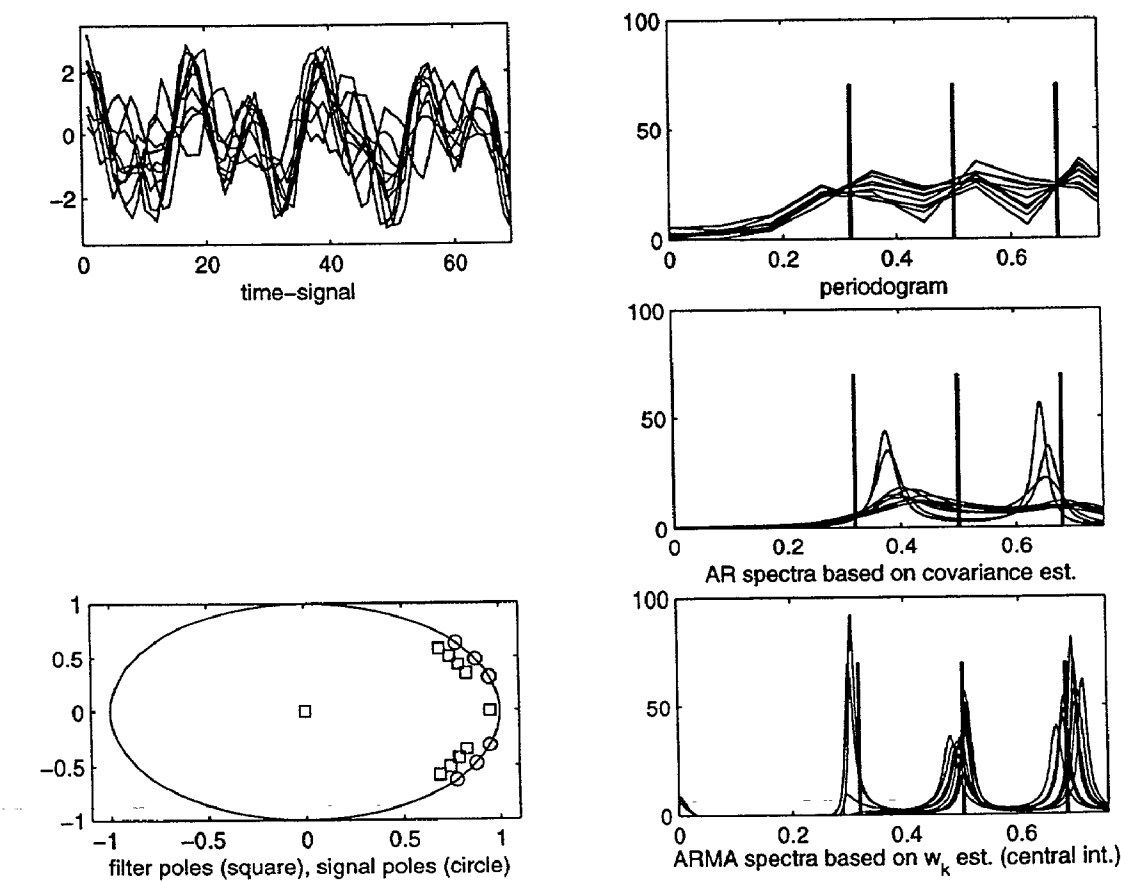
Figure 3: Resolution of spectral lines.

8   C. I. BYRNES, T. T. GEORGIOU, AND A. LINDQUIST

the periodogram, while subplots (2,2) and (3,2) show the respective spectra obtained
using methods (ii) and (iii) respectively. In each case, lines marking the frequencies of
the sinusoids were drawn for comparison. Finally, subplot (3,1) shows the frequencies
of the sinusoids (marked by "o" on the unit circle) along with the selected poles for
the filter bank (marked by " "). The superiority of our method is quite evident.

In conclusion, the bank of filters is one of the three basic elements in our method
for spectral estimation and filtering. The next fundamental ingredient is a complete
parameterization of all positive-real functions of a degree at most n which satisfy
n such interpolation constraints. This is a Nevanlinna-Pick problem with degree
constraint. The constraint makes the problem *nonlinear* in general and the selection of
particular solutions requires the third ingredient of our theory, a convex optimization
approach based on the concept of generalized entropy, which is explained in Section
4. The next two sections contain the relevant facts.

3. Nevanlinna-Pick interpolation

Analytic interpolation theory has its roots in classical mathematics going back to
the end of the last century, on approximate integration, quadrature formulae and the
moment problem. The foundations, of what were to become known as the Nevanlinna-
Pick problem were laid out by C. Carathéodory, I. Schur, R. Nevanlinna, G. Pick, G.
Szegö in the beginning of this century; see, e.g., [22, 34]. The subject evolved into a
rich topic in operator theory [31, 29, 16].

3.1. The classical theory. The basic question of the *Nevanlinna-Pick problem* is
as follows: Given a set of pairs (z_k, w_k), $k = 0, 1, 2, \ldots, n$, with z_k's distinct and with
$|z_k| > 1$, is it possible to construct a positive-real function $f(z)$ such that

$$f(z_k) = w_k \quad k = 0, 1, \ldots, n?$$

In case a solution exists, the problem consists in parameterizing all solutions.
    The solvability condition is expressed in terms of the so-called *Pick matrix*

$$P_n := \left[\frac{w_k + \bar{w}_\ell}{1 - \bar{z}_k^{-1} z_\ell^{-1}} \right]_{k,\ell=0}^n . \tag{3.1}$$

In particular, the Nevanlinna-Pick problem admits a solution if and only if P_n is non-
negative definite. In case $P_n \geq 0$ but singular the solution is unique. In case $P_n > 0$,
the complete set of solutions is given by a linear fractional transformation, which is
constructed from the interpolation data, acting on a "free" parameter function which
is only required to have certain analytic properties, e.g., to be positive-real. A detailed
exposition can be found in [34].
    A generalization of the problem known as the Carathéodory-Fejér problem, allows
for the possibility that $f(z)$ is specified both in terms of values and derivatives up to
some order at points z_i outside the disc. Again, the solvability condition is expressed
in terms of a suitable (generalized) Pick matrix and all solutions are parameterized by
a linear fractional transformation. We refer the reader to the standard mathematics
literature [34, 22, 31, 29] for details.

3.2. Interpolation with a degree constraint. While the classical results are quite elegant, the parameterization of all solutions to the interpolation problem includes functions which may have very high degree or even be nonrational. In engineering applications, the *interpolant* $f(z)$ needed to be of bounded degree. In fact, it may be the transfer function of a finite-dimensional dynamical system, e.g., the driving point impedance of a network [35], or relate to a Gauss-Markov model for a process (see Appendix A). Such considerations raised the question of identifying minimal degree solutions to analytic interpolation problems ([35] and later in [25]).

When solutions exist, then there is a particular one which has degree $\leq n$, namely the so-called *central solution* corresponding to a trivial choice for the free parameter. However, for all practical purposes, the question of determining the minimal degree interpolants is still open [17, 7].

However, the question of identifying and parameterizing all solution of degree $\leq n$ is more fruitful and has led to a rich theory, [3]–[11] and [17]–[20], which provides us with the following complete parameterization of all such solutions.

Theorem 3.1. *Given a set of interpolation points* $z_0, z_1, \ldots, z_n$ *outside the unit circle, and a set of corresponding values* $w_0, w_1, \ldots, w_n$ *such that the Pick matrix* P_n *in* (3.1) *is positive definite and an arbitrary pseudo-polynomial*

$$d(z) = d_n z^n + \cdots + d_1 z + 1 + \bar{d}_1 z^{-1} + \ldots \bar{d}_n z^{-n}, \qquad (3.2)$$

which is non-negative on the unit circle , i.e., for $z = e^{i\theta}$, *then there exists a* <u>*unique*</u> *pair of polynomials*

$$a(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n \quad and \quad b(z) = b_0 z^n + b_1 z^{n-1} + \cdots + b_n,$$

of degree at most n *and without roots in* $|z| > 1$, *such that*

$$d(z) \;=\; \bar{a}(z^{-1})b(z) + \bar{b}(z^{-1})a(z), \qquad (3.3)$$

$$\frac{b(z)}{a(z)} \;=: \; f(z) \quad is\ a\ positive\text{-}real\ function,\ and \qquad (3.4)$$

$$f(z_k) \;=\; w_k \quad k = 0, 1, \ldots, n. \qquad (3.5)$$

This theorem also holds in the more general case where the interpolation data are of the Carathéodory-Féjer type, i.e., include constraints on the derivative of $f(z)$, and was first discussed in the special (Carathèodory) case with a single multiple interpolation point at $z = \infty$, the so-called rational covariance extension problem. Existence was first proven in this context in [17, 19] and uniqueness, as well as well-posedness, in [6]; see [10, 8, 9] for alternative proofs. Existence for the distinct point Pick-Nevaninna problem was proven in [18] and uniqueness in [20] and, for strictly positive real functions, in [11]. The theorem extends to interpolation of matrix-valued functions (see [17] where existence of solutions were shown in the context of Carathèodory interpolation). An approach generalizing this result to the context of the commutant-lifting theory is the subject of [12].

An additional point to be noted is that in case the set of interpolation pairs is closed under complex conjugation, and d has real coefficients, then $f(z)$ is a real function [11]. In fact, this is the case of interest in the current paper. In subsequent sections,

without loss of generality, we will make an additional simplifying assumption, namely that

$$z_0 = \infty.$$

This is done for notational convenience and is often dictated by practical considerations in the application of the theory. For example, in the context of Section 2, the trivial filter $G_0(z) \equiv 1$ is a natural first choice.

The requirement that the degree of the interpolant be at most n imposes a nonlinear constraint on the class of solutions. Theorem 3.1 states that each choice of a nonnegative trigonometric polynomials (3.2) corresponds to exactly one solution to this *nonlinear Nevanlinna-Pick problem* and that it is defined by (3.4)–(3.5) and the degree constraint. Moreover all such solution is generated in this way.

We have seen in Section 2 how positive-real functions relate to the power spectrum of a stochastic process and how interpolation data can be obtained by covariance estimates of the output of a filter bank. The choice of the *spectral zeros* of the process, i.e., the roots of $d(z)$ located in the unit disc, is an important design variable. For instance, in lack of specific information, one may be looking for an all-pole model (i.e., AR-model) for the process, which corresponds to $d(z) = 1$. Nevertheless, even in this case, for general interpolation data (i.e., which are not necessarily at ∞), traditional techniques based on the Levinson algorithm are not applicable. This is due to the fact that the interpolation data are no longer in the form of a partial covariance sequence. Furthermore, the classical Nevanlinna-Pick theory and the linear-fractional transformation parameterizing all solutions, is not applicable either. The reason being that the transformation does not allow inference about the degree of the interpolant, except in the special case of the central solution. However, the central solution does not produce an AR model but a model with spectral zeros precisely at the interpolation points z_k^{-1}.

4. Generalized entropy and convex optimization

In this section we describe how an arbitrary solution of the Nevanlinna-Pick interpolation problem, as described in Theorem 3.1, can be obtained from a convex optimization problem, and we summarize the steps of a numerical algorithm based on this optimization problem. The complete theory has been developed in [11].

As in Theorem 3.1, we we start with

(i) a set of distinct interpolation points $z_0, z_1, \ldots, z_n$ such that $|z_k| > 1$ and a set of corresponding set of interpolation points $z_0, z_1, \ldots, z_n$ such that the Pick matrix (3.1) is positive definite. Throughout we assume that $z_0 = \infty$.

(ii) $d(z)$ a non-negative trigonometric polynomial (3.2) of degree n.

The assumption that $z_0 = \infty$ is without loss of generality. It is a convenient normalization which, in subsequent applications of the theory, represents an appropriate choice.

We need some additional notation and basic facts which we now review.

4.1. Notation and basic facts. Given the interpolation points $z_0, z_1, \ldots, z_n$, define the polynomial

$$\tau(z) := \prod_{k=1}^{n} (z - \bar{z}_k^{-1}) = z^n + \tau_1 z^{n-1} + \ldots + \tau_{n-1} z + \tau_n, \qquad (4.1)$$

. _ _

A NEW APPROACH TO SPECTRAL ESTIMATION   11

and consider the all-pass function

$$B(z) := z^{-1}\frac{1 + \bar{\tau}_1 z + \cdots + \bar{\tau}_{n-1}z^{n-1} + \bar{\tau}_n z^n}{z^n + \tau_1 z^{n-1} + \cdots + \tau_{n-1}z + \tau_n} = z^{n-1}\frac{\tau^*(z)}{\tau(z)}, \qquad (4.2)$$

where $\tau^*(z) := \bar{\tau}(z^{-1})$. Such an all-pass function is called a (finite) Blasckhe product. Next define

$$H(B) := H_2 \ominus BH_2,$$

where H_2 denotes the Hardy space of functions which are analytic outside the unit disc and have square-integrable radial limits. Functions in H_2 are precisely the Fourier transforms of square-integrable discrete-time functions which vanish for negative times. Next, let L_2 denote the Hilbert space of functions which are square-integrable on the unit circle, and as usual, H_2 is identified with a subspace of L_2. It is easy to prove that the space $H(B)$ is of dimension $n + 1$. Indeed, the elements of the filter bank of Section 2.3,

$$G_k(z) = \frac{z}{z - p_i} \quad \text{with } p_k = \bar{z}_k^{-1} \text{ for } k = 0, 1, \ldots, n,$$

can be shown to form a basis for $H(B)$. These basis elements are also known as *Cauchy kernels* and satisfy the *Cauchy formulae*:

$$\langle f, G_k \rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} \bar{G}_k(e^{-i\theta})f(e^{i\theta})d\theta = f(z_k), \quad f \in H_2, \qquad (4.3)$$

$$\langle f^*, G_k \rangle = \overline{f(\infty)}, \qquad (4.4)$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in L_2. It follows that any function $a(z) \in H(B)$ is of the form

$$a(z) = \sum_{i=0}^{n} \alpha_i G_i(z) = \frac{a_0 z^n + \ldots + a_n}{\tau(z)}. \qquad (4.5)$$

Finally, define the space of quasi-polynomials

$$\mathcal{S} := \{Q \mid Q(z) = \sum_{k=0}^{n}(q_k G_k + \bar{q}_k G_k^*)\},$$

and the convex subset of (strictly) positive ones:

$$\mathcal{S}_+ := \{Q \mid Q \in \mathcal{S} \quad \text{and} \quad Q(e^{i\theta}) > 0 \text{ for all } \theta \in (-\pi, \pi]\} \subset \mathcal{S}.$$

4.2. Entropy functionals and convex optimization. Now, given the polynomial (4.1) and the pseudo-polynomial (3.2), define the function

$$\Psi(z) := \frac{d(z)}{\tau^*(z)\tau(z)}. \qquad (4.6)$$

Then, for any positive real function f, the functional

$$\psi(f) := \langle \log(f + f^*), \Psi \rangle, \qquad (4.7)$$

represents a generalized entropy gain which plays a key role in our theory. In fact, in [11, Theorem 4.1] we have the following result.

12   C. I. BYRNES, T. T. GEORGIOU, AND A. LINDQUIST

Theorem 4.1. *Given any* $\Psi \in \mathcal{S}_+$ *there exists a unique solution to the constrained optimization problem*

$$\max\{\ _\Psi(f) \mid f \text{ is positive-real, } f(z_k) = w_k \text{ for } k = 0, 1, \ldots, n\}. \qquad (4.8)$$

Furthermore, this solution is of the form

$$f(z) = \frac{b(z)}{a(z)}, \ a, b \in H(B), \qquad (4.9)$$

and

$$a^*(z)b(z) + b^*(z)a(z) = \Psi(z). \qquad (4.10)$$

Conversely, if f *is positive-real, satisfies the interpolation conditions, and the above two equations (4.9)–(4.10), then it is the unique solution to (4.8).*

Note that (4.9) is equivalent to requiring that f is of degree at most n. The choice $\Psi = 1$ yields the central solution of the Nevanlinna-Pick theory which is also known as the "maximum entropy" solution. All other interpolants of degree $\leq n$ can be obtained by choosing the corresponding Ψ and solving the "generalized entropy" maximization problem given above. However, this optimization problem is infinite-dimensional, and hence not easy to solve. As it turns out, it has a dual with finitely many variables, and next we shall turn to this problem.

To this end, let $w(z)$ be a function, analytic in $\{z \mid |z| > \epsilon\}$ for some $\epsilon \in (0, 1)$, satisfying the interpolation condition

$$w(z_k) = w_k, \quad k = 0, 1, \ldots, n, \qquad (4.11)$$

and define for each function $Q \in \mathcal{S}_+$ the functional

$$_\Psi(Q) := \langle Q, w + w^* \rangle - \langle \log Q, \Psi \rangle. \qquad (4.12)$$

It can be shown that the linear term $\langle Q, w + w^* \rangle$ does not depend on the particular choice of w but only on the interpolation data. In fact, if $Q(z) = a(z)a^*(z)$ with $a(z) \in H(B)$ as in (4.5), then

$$\langle Q, w + w^* \rangle = (\alpha_0 \ldots \alpha_n) P \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_n \end{pmatrix}, \qquad (4.13)$$

with P being the Pick matrix defined by (3.1). For reasons which will become clear shortly, we now fix w to be the unique function in $H(B)$ satisfying (4.11). This w is given by

$$w(z) = w_0 + \frac{\pi_1 z^{n-1} + \pi_2 z^{n-2} + \cdots + \pi_n}{\tau(z)}$$

where $\pi_1, \pi_2, \ldots, \pi_n$ satisfies the Vandermonde system

$$\begin{bmatrix} z_1^{n-1} & z_1^{n-2} & \ldots & 1 \\ z_2^{n-1} & z_2^{n-2} & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ z_n^{n-1} & z_n^{n-2} & \ldots & 1 \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_n \end{bmatrix} = \begin{bmatrix} (w_1 - w_0)\tau(z_1) \\ (w_2 - w_0)\tau(z_2) \\ \vdots \\ (w_n - w_0)\tau(z_n) \end{bmatrix}.$$

A NEW APPROACH TO SPECTRAL ESTIMATION 13

Since the points $z_0, z_1, \ldots, z_n$ are distinct, this system has a unique solution. Note that w is not positive real in general, and therefore cannot be used as an interpolant, but it will play an important role in the sequel.

Using duality theory, the maximization problem of Theorem 4.1 can be seen to be equivalent to the following convex optimization problem; see [11, Theorem 4.5].

Theorem 4.2. *Given Φ and w as above, there exists a unique solution to the following problem:*

$$\min\{ \, {}_{\Psi}(Q) \mid Q \in \mathcal{S}_+\}. \qquad (4.14)$$

Moreover, for the minimizing Q, there is a unique positive real function f satisfying the interpolation condition $f(z_k) = w_k$ for $k = 0, 1, \ldots, n$ such that

$$\frac{\Psi_0(z)}{Q(z)} = f(z) + f^*(z),$$

(4.9) and (4.10) hold. Conversely, any positive-real function with these properties arises as the unique solution of (4.14).

Theorem 4.1 and Theorem 4.2 each imply Theorem 3.1 in the case where Ψ has no roots on the circle. (The part of Theorem 3.1 pertaining to Ψ having roots on the circle is proved in [20] by a nonconstructive argument.) The advantage with the present proof is that it is constructive and therefore yields an algorithm for computing an arbitrary interpolant of degree at most n. Since Q is determined by $n + 1$ variables $q_0, q_1, \ldots, q_n$, Theorem 4.2, it is a finite-dimensional optimization problem.

The proofs of Theorem 4.1 and Theorem 4.2, which are very nontrivial, are given in [11]. Since $_{\Psi}$ is a strictly convex function on a convex set $\mathcal{S}_+$, the minimization problem of Theorem 4.2 is a convex optimization problem. Therefore, if there is a minimum in the the open set $\mathcal{S}_+$, this minimum is unique and occurs at a stationary point, i.e., at a point where the gradient is zero. It is proved in [11] that this is indeed the case. It is then quite straight-forward to show that the optimal Q defines a unique interpolant f with the required properties. Since this is quite instructive, we give an alternative proof of this, tailored to our present exposure in Appendix B. Elements for this from will be needed to derive the gradient of $_{\Psi}$ needed to solve the convex optimization problem.

4.3. Computations in state-space. We now outline the algorithm provided by Theorem 4.2 in the context of state-space representations. A gradient method is used to obtain the minimum of $_{\Psi}(Q)$.

Any function $a(z) \in H(B)$ can be completely specified by its values at any set of $n + 1$ distinct points in the domain of analyticity (i.e., in $\{z : |z| > 1\}$). It is also specified by its value, and the value of its first n derivatives at ∞. This gives rise to the following alternative representations:

$$
\begin{aligned}
a(z) &= \sum_{i=0}^{n} \alpha_i G_i(z) = \frac{a_0 z^n + \ldots + a_n}{\tau(z)} \\
&= h_0 + h_1 z^{-1} + \ldots + h_n z^{-n} + \ldots \qquad (4.15) \\
&= d + c(zI - A)^{-1} b,
\end{aligned}
$$

14   C. I. BYRNES, T. T. GEORGIOU, AND A. LINDQUIST

where

$$
\left[\begin{array}{c|c} A & b \\ \hline c & d \end{array}\right] = \left[\begin{array}{ccccc|c} -\tau_1 & -\tau_2 & \ldots & -\tau_{n-1} & -\tau_n & h_n \\ 1 & 0 & \ldots & 0 & 0 & h_{n-1} \\ 0 & 1 & \ldots & 0 & 0 & h_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & h_1 \\ \hline 0 & 0 & \ldots & 0 & 1 & h_0 \end{array}\right].
\tag{4.16}
$$

The last representation is especially convenient, and hence we adopt this notation with (A, c) universal for all elements in $H(B)$, and the vector of *Markov coordinates*

$$
\begin{bmatrix} b_a \\ d_a \end{bmatrix} := \begin{bmatrix} h_n \\ \vdots \\ h_1 \\ h_0 \end{bmatrix}
$$

then completely determines $a(z) \in H(B)$. We write $a(z) \sim (b_a, d_a)$ to denote this correspondence. Elements in $\mathcal{S}_+$ can be represented by their analytic part which belongs to $H(B)$, e.g., if $Q \in \mathcal{S}_+$ then $Q = q + q^*$ with $q \in H(B)$.

Applying a gradient method, or, if needed, Newton's method, to the convex optimization problem of Theorem 4.2, we obtain a recursive algorithm. In particular, start with a positive real $q \in H(B)$, e.g., $q \equiv 1$. Then iterate through the following steps.

0. Given a positive-real function $q \in H(B)$:
1. Compute a "minimum-phase" $a \in H(B)$ such that $aa^* = q + q^*$.
2. Compute $b \in H(B)$ such that $a^*b + b^*a = \Psi$.
3. Compute $f = b/a$.
4. Evaluate the gradient $\nabla_q \Psi$.
5. If $\|\nabla_q \Psi\|$ is greater than a prespecified threshold,   then update $q \to q_{\text{new}} = q - \epsilon\nabla_q \Psi$ and return to Step 1.

The choice $q(z) \equiv 1$ can always be taken as a starting point in $H(B)$. The value $\epsilon > 0$ should be selected small enough for q_{new} to be positive-real. The algorithm can be modified to include the Hessian matrix $\nabla_{qq} \Psi$, the update in last step can be performed by, e.g., Newton's method.

We now outline the algorithmic steps using state-space representations.

Step 1. Given $q(z) \sim (b_q, d_q)$, the spectral factorization $aa^* = q + q^*$ is standard (cf. [11]), and can be computed by solving the (discrete-time) algebraic Riccati equation:

$$
P - APA' - (b_q - APc')(2d_q - cPc')^{-1}(b_q - APc')' = 0.
$$

Then $a(z) \sim (b_a, d_a)$, the minimum-phase spectral factor in $H(B)$ of $Q = q + q^*$, is given by

$$
\begin{aligned}
d_a &= \sqrt{2d_q - cPc'} \\
b_a &= (b_q - APc')/d_a.
\end{aligned}
$$

Step 2. Given $a(z) \sim (b_a, d_a)$ and $\Psi(z) =: \psi(z) + \psi^*(z)$ with $(b_\psi, d_\psi) \sim \psi \in H(B)$, solving $a^*b + b^*a = \Psi$ with $\Psi \in \mathcal{S}_+$ for $b(z)$ is equivalent to solving a set of linear equations. It is shown in [11] that this amounts to solving the Lyapunov equation

$$P_1 - A_1 P_1 A_1' = Q_1$$

where

$$
\begin{aligned}
A_1 &= A - b_1 c \\
b_1 &= b_a/d_a \\
b_2 &= b_\psi/(2d_\psi) \\
Q_1 &= 2d_\psi [b_2 b_2' - (b_2 - b_1)(b_2 - b_1)'],
\end{aligned}
$$

to obtain $b(z) \sim (b_b, d_b)$ via

$$
\begin{aligned}
d_b &= (d_\psi - \tfrac{1}{2} c P_1 c')/d_a \\
b_b &= (b_\psi - A P_1 c' - b_a d_b)/d_a.
\end{aligned}
$$

Step 3. Given $a(z) \sim (b_a, d_a)$ and $b(z) \sim (b_b, d_b)$, both in $H(B)$, the fraction $f = b/a$ is no longer in $H(B)$. However, since $a(z)$ is minimum-phase, i.e., has no roots outside the disc, $f(z)$ is analytic outside the disc. The important thing in this step is to cancel the common denominators of a and b, and standard algebraic manipulations yield the realization

$$
f \sim \left[\begin{array}{c|c} A_f & b_f \\ \hline c_f & d_f \end{array} \right] = \left[\begin{array}{c|c} A - b_a c/d_a & -b_a \frac{d_b}{d_a^2} + b_b \frac{1}{d_a} \\ \hline c & \frac{d_b}{d_a} \end{array} \right]
$$

of the same order as the common degree of a and b. Because $f \notin H(B)$, A_f differs from A.

Step 4. Since $f \notin H(B)$, we begin by projecting f onto $H(B)$ to obtain an orthogonal projection of the form

$$\hat{f}(z) := f(z_0) + \frac{\sigma(z)}{\tau(z)}, \quad \text{where } \sigma(z) = \sigma_1 z^{n-1} + \cdots + \sigma_n.$$

This is done geometrically in [11], but, since $\hat{f}(z) = f(z) + B(z)g(z)$ for some $g \in H_2$ (see [11]), we have

$$\hat{f}(z_k) = f(z_k) \quad k = 0, 1, \ldots, n,$$

and consequently we can use the same procedure as on page 12 to determining the unique solution $\sigma_1, \sigma_2, \ldots, \sigma_n$ of the Vandermonde system

$$
\begin{bmatrix}
z_1^{n-1} & z_1^{n-2} & \cdots & 1 \\
z_2^{n-1} & z_2^{n-2} & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
z_n^{n-1} & z_n^{n-2} & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
\sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_n
\end{bmatrix}
=
\begin{bmatrix}
[f(z_1) - f(z_0)] \tau(z_1) \\
[f(z_2) - f(z_0)] \tau(z_2) \\
\vdots \\
[f(z_n) - f(z_0)] \tau(z_n)
\end{bmatrix}.
$$

Then, given $\hat{f} \sim (b_{\hat{f}}, d_{\hat{f}})$ and $w \sim (b_w, d_w)$, the gradient, represented as a column vector, is

$$\nabla_q \ \Psi = 2 \begin{bmatrix} P_2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} b_w - b_{\hat{f}} \\ d_w - d_{\hat{f}} \end{bmatrix}, \tag{4.17}$$

where P_2 is the unique solution to the Lyapunov equation

$$P_2 = A'P_2A + c'c,$$

and "0" denotes a suitable row or column of zeros. The proof of this is given in Appendix B.

Step 5. The step ϵ should be sufficiently small so that $q - \epsilon\nabla_q \, \psi$ is positive real. Following our convention of using Markov coordinates for representing functions in $H(B)$,

$$q - \epsilon\nabla_q \, \psi \sim \begin{bmatrix} b_q \\ d_q \end{bmatrix} - 2\epsilon \begin{bmatrix} P_2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} b_w - b_f \\ d_w - d_f \end{bmatrix}$$

This corresponds to a positive real function if Step 1 above produces a positive definite P. A rule of thumb is to choose ϵ so that the change in the values of the Markov parameters of q initially is less than 1%. Testing how close the roots of $a(z)$ are from to unit circle can be used to increase/decrease ϵ in successive steps. The loop is completed by returning to Step 1, after testing whether $\|\nabla_q \, \psi\|$ is still large. For instance, we may be chose to terminate the algorithm, if, say, the change is less than 0.01%.

4.4. Central solution. The special case when $\Psi = 1$ in Theorem 4.1 is much simpler, and is the only case for which there has been computational procedures so far. Although it can be obtained using the computational approach of Section 4.3, the problem is in fact linear. Therefore the solution can be derived by direct algebraic computations. For the purposes of our present paper we include in Appendix C an outline of the relevant theory, in a slightly new form, and summarize below the formula for this so-called central interpolant $f(z)$.

Let $\hat{B} := \frac{\tau^*(z)z^n}{\tau(z)}$, and let $\hat{w}(z)$ be the unique function in $H(\hat{B})$ which interpolates

$$\hat{w}(z_k) = w_k/w_0 \quad \text{for } k = 1, 2, \ldots, n. \tag{4.18}$$

(Note the that $\hat{B}(z)$ differs from our earlier $B(z)$ by a factor of z^{-1} for reasons explained in Appendix C. Also the expression for elements in $H(\hat{B})$ given above differs from a similar expression for elements in $H(B)$ used earlier.) Equation (4.18) gives rise to n linear equations in the coefficients of $\hat{w}(z)$, analogous to the ones on page 12. Compute a state-space realization (a, b, c, d) for

$$\begin{bmatrix} \hat{w} & \hat{B} \\ 0 & 0 \end{bmatrix} = c(zI - a)^{-1}b + d.$$

Let P be the solution of (C.3) obtained by iterating (C.5). Perform a singular value decomposition of $\mathcal{M}$ given in (C.4), let U be the matrix of singular vectors as in (C.6), and determine c_+, d_+ by multiplying the first two columns of U with $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ on

the right. Then, define a_L, b_L, c_L, d_L to be as in (C.7), and set

$$
x \ := \ -\left([1, \ -1]d_L \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \Big/ \left([1, \ -1]d_L \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)
$$

$$
b_0 \ := \ b_L \begin{bmatrix} 1 + x \\ 1 - x \end{bmatrix}, \ \ d_0 := d_L \begin{bmatrix} 1 + x \\ 1 - x \end{bmatrix}
$$

$$
c_1 \ := \ [1, \ 0]c_L, \ \ c_2 := [0, \ 1]c_L
$$

$$
d_1 \ := \ [1, \ 0]d_0, \ \ d_2 := [0, \ 1]d_0.
$$

Finally, then,

$$
\left[\begin{array}{c|c} a_f & b_f \\ \hline c_f & d_f \end{array}\right] = \left[\begin{array}{c|c} a_L - b_0 d_1^{-1} c_1; & \frac{w_0}{d_1} b_0 \\ \hline c_2 - \frac{d_2}{d_1} c_1 & \frac{w_0}{d_1} d_2 \end{array}\right]
$$

is the required state-space representation of $f(z)$.

5. Case studies

First we discuss certain low order examples. We compare standard covariance-based AR filtering, with AR or ARMA filtering based on generalized interpolation as to how well they approximate a given spectrum. Next, we consider a high-order example and repeat a similar analysis. Finally, we study, by simulation, the reliability of covariance as well as general interpolation estimates, and their effect on the spectrum. This complements our earlier simulation study which was presented as a motivating example in Section 2.

5.1. Low order example. The process y is generated by a "nominal" ARMA(2,1) filter driven by unit variance white noise. The filter has zeros/poles at

Zeros	Poles	
$.9e^{\pm\frac{i\pi}{3.2}}$	$.9e^{\pm i\pi/3}$	$.3e^{\pm i\pi/3.5}$

Accordingly, the nominal positive real function is computed as

$$
f_y(z) = 0.5569 \frac{z^4 - 0.7901z^3 + 0.4184z^2 + 0.3487z - 0.0729}{z^4 - 1.2741z^3 + 1.2367z^2 - 0.3840z + 0.0729}.
$$

Subplots $(1, 1)$ and $(2, 1)$ in Figure 4 show the pole-zero pattern for the nominal spectrum and the spectral density function $\Phi_y(e^{i\theta})$, respectively. The first four covariance lags are computed as follows:

Covariance values
$c_0 = 1.1138, \ c_1 = 0.2696, \ c_2 = -0.1123, \ c_3 = -0.0683, \ c_4 = 0.0741.$

The AR model for y, corresponding to these covariance samples, has poles (marked by "$\times$") shown in subplot $(1, 2)$ and spectral density function shown in subplot $(2, 2)$.
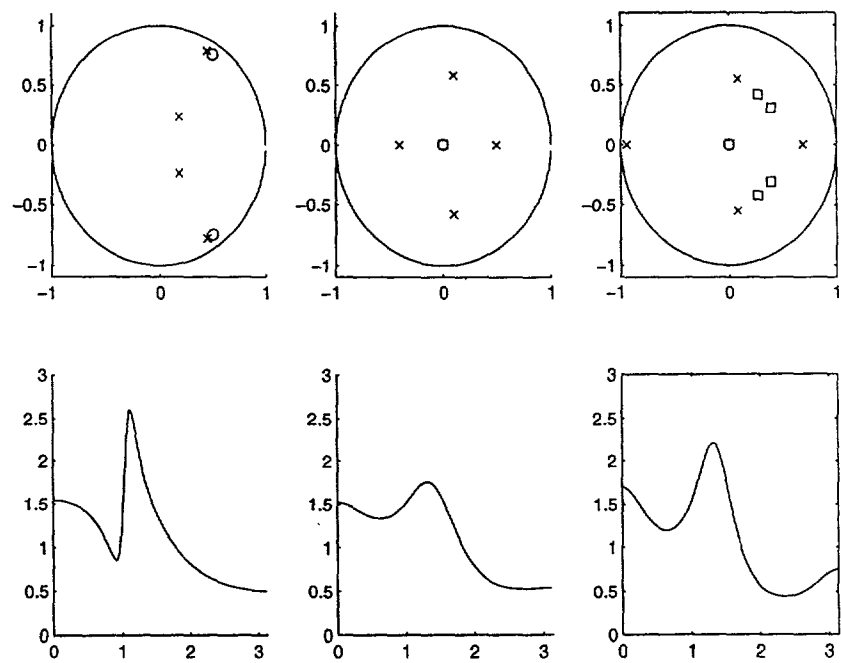
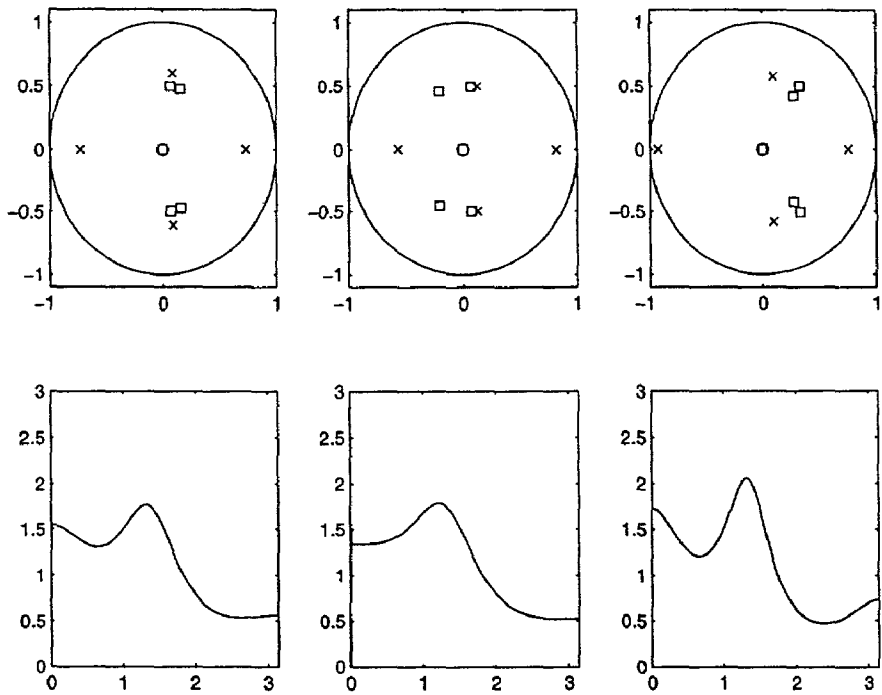Figure 4: AR spectra based on covariance data and interpolation data vs. the exact spectrum.

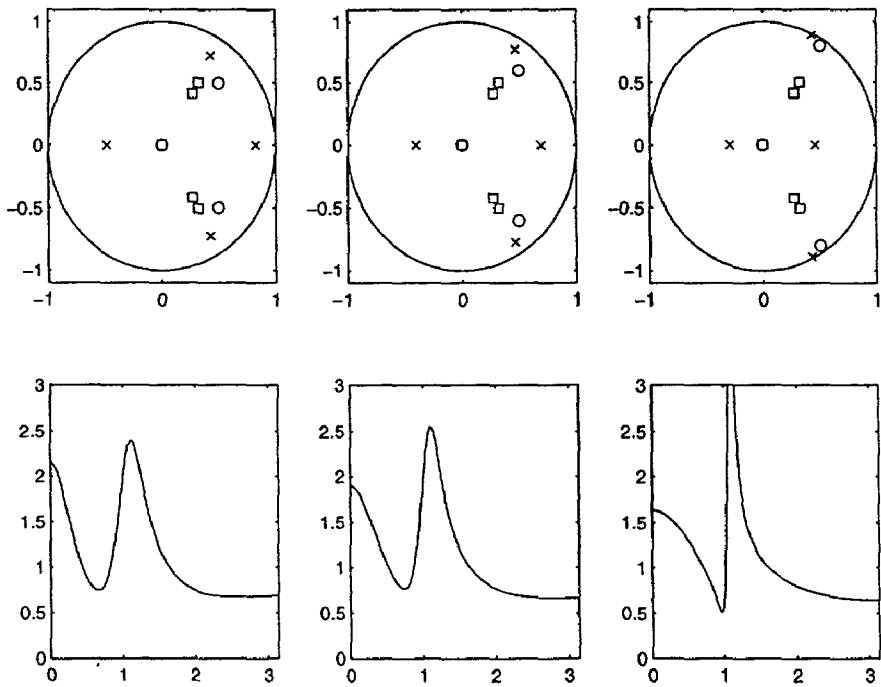Figure 5: AR modeling from interpolation data.

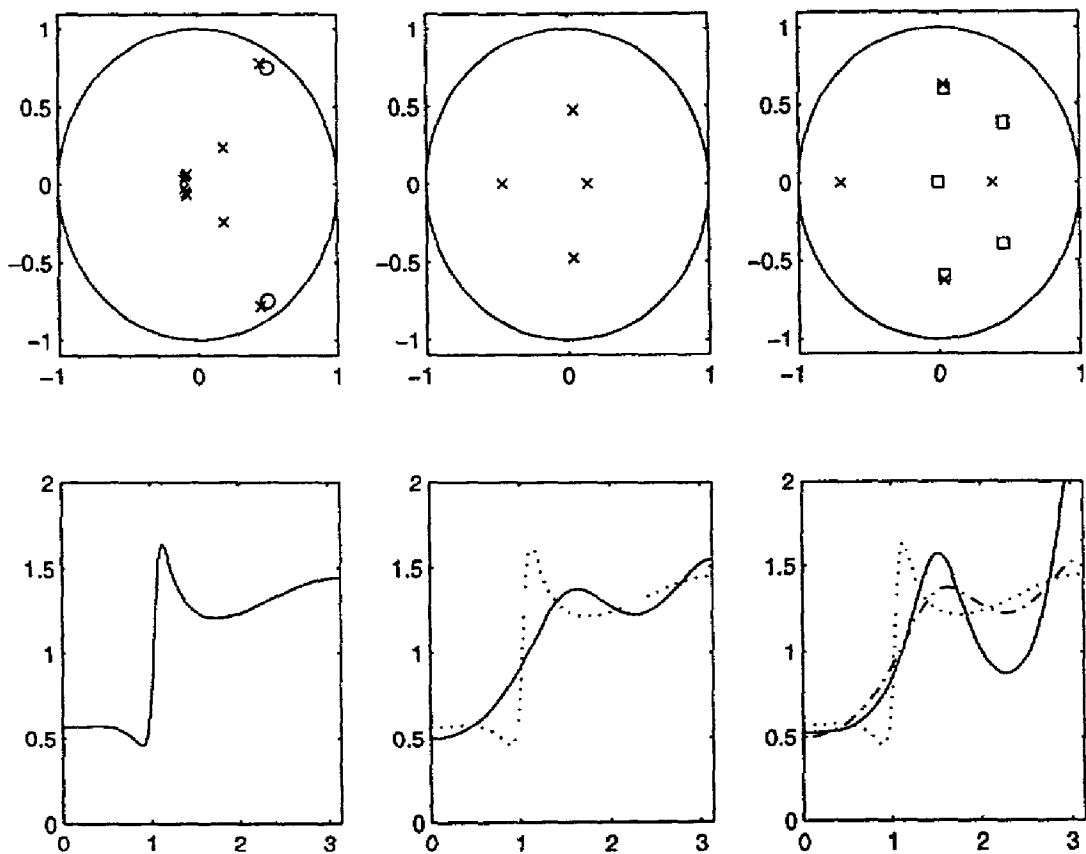Figure 6: ARMA modeling from interpolation data.

Figure 7: Higher-order case.

106

The quantities used to produce the plots shown in Figure 7 are as follows:

$$\Phi_y(e^\theta) = |\frac{z - .9e^{i\pi/3.2}}{\Pi_{k=1}^5(z - \rho_k)}|^2,$$

with $\rho_0, \rho_1, \ldots, \rho_4$ taking the values $0.9e^{i\pi/3}$, $0.3e^{i\pi/3.5}$, $0.1e^{i\pi/1.1}$, $0.1e^{i\pi/1.2}$, and $0.1e^{i\pi/1.3}$, respectively. The points $p_k = z_k^{-1}$ where $f_y(z)$ is evaluated are taken to be at 0, $0.6e^{\pm.7i}$, and $.6e^{\pm 1.5i}$.

The three positive real functions $f_y(z)$, $f_{AR}(z)$, and $f(z)$, where $f_y(z)$ is the nominal one, $f_{AR}(z)$ is the one corresponding to an AR-model matching the first five covariance samples, and $f(z)$ is the one corresponding an AR-model but matching the values of $f_y(z_k^{-1})$ ($k = 0, 1, \ldots, 4$) instead, were computed to have poles/zeros tabulated below:

$f_y(z)$	at ∞ 0.5409	$f_{AR}(z)$	at ∞ 0.5409	$f(z)$	at ∞ 0.5408
poles	zeros	poles	zeros	poles	zeros
$.4500 \pm .7794i$	$.4995 \pm .6746i$	$.0426 \pm .4767i$	$-.2665 \pm .3337i$	$.0386 \pm .6287i$	$-.3752 \pm .5133i$
$.1870 \pm .2345i$	$.0242 \pm .2617i$	$-.4583$	$.5504$	$.3877$	$.4775 \pm .1890i$
$-.0749 \pm .0663i$	$-.0657 \pm .0366i$	$.1419$	$.1484$	$-.6932$	
$-.0959 \pm .0282i$	$-.0566 \pm .0713i$				
$-.0866 \pm .0500i$	$-.0747$				
	$.4279$				

5.3. Simulation results. In order to demonstrate typical tradeoffs between different choices of interpolation points, variability and resolution of estimated spectrum, we present results from a simulation study. A Gaussian stochastic process with power spectrum

$$\Phi_y(z) = |\frac{z^2 - z + .81}{z^4 - 1.27z^3 + 1.24z^2 - .38z + .07}|^2,$$

having spectral zeros at $.9e^{\pm i\pi/3.2}$ and stable poles at $.9e^{\pm i\pi/3}$, $.3e^{\pm i\pi/3.5}$, was simulated in Matlab. We ran the simulation 30 times. Each time, we generated $N = 200$ values of the process, and out of those values we estimated the power spectrum in the following two ways:

(a) The first five covariance lags and the corresponding AR spectrum was estimated.

(b) The covariances at the output of a filter bank and the ARMA spectral estimate corresponding to the central solution was determined. The poles of the filter-bank were chosen at $\{0, .5e^{\pm i}, .5e^{\pm i/1.5}\}$.

The tuning of the filter bank was dictated by the desire to obtain better resolution in the vicinity of the spectral peak. Thus, the poles were "approximately clustered" in the neighborhood of the targeted "ark". Their distance from the circle was selected so that the variance of the estimated values was "reasonable".

The subplots in Figure 8 display the following:

(1,1) Poles ($\times$), spectral zeros ($\circ$), and the choice of filter-bank poles $p_k = z_k^{-1}$, $k = 0, 1, \ldots, 5$, marked with " ".

(1,2) Mean values ($\circ$), and the one-standard-deviation interval from the mean, for each of the first five covariance samples $\hat{c}_k$, $k = 0, 1, \ldots, 5$, for the process.
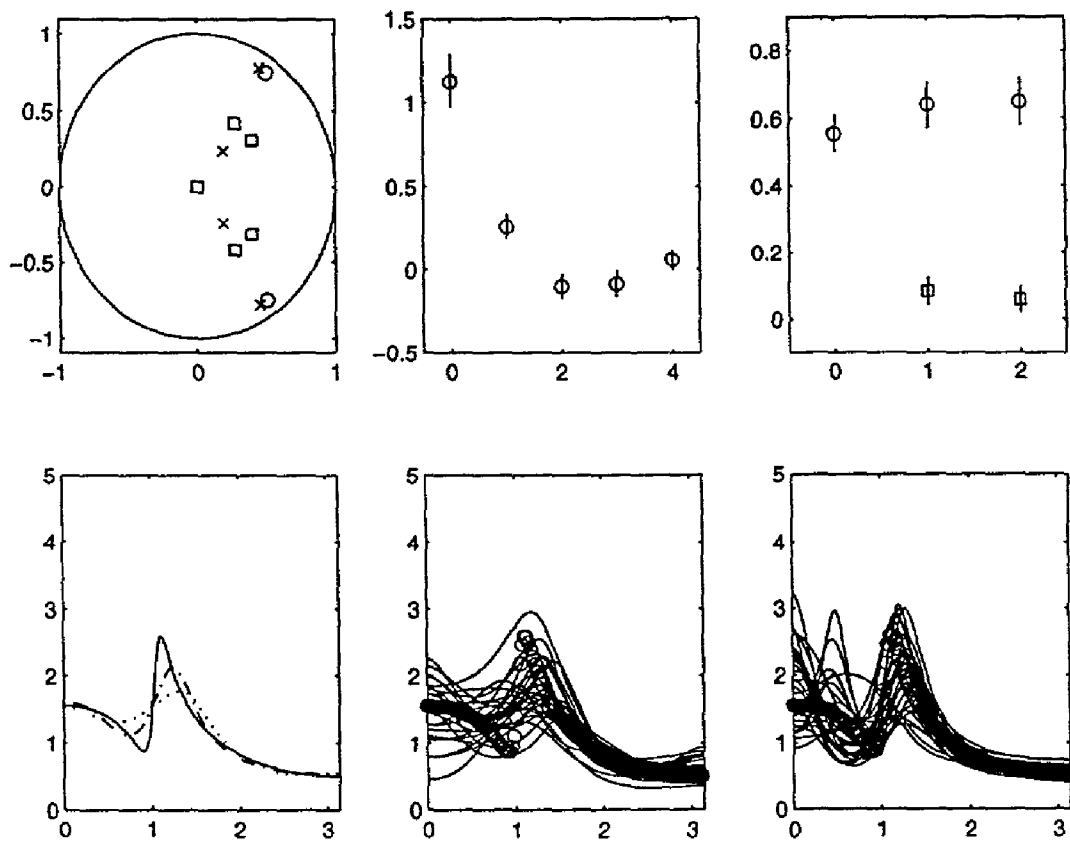
107

Figure 8: Simulation study.

(iii) A convex-optimization approach for computing any arbitrary solution described the theory in (ii). The algorithmic steps are given in state-space form.

THREE can be tuned to give improved resolution at selected portions of the spectrum. More specifically, we have demonstrated that selection of the filter-bank poles in the vicinity of any arc of the unit circle results in improved reproduction of the power spectrum in the corresponding frequency band, as compared to, e.g., traditional AR filtering. Another tunable set of parameters are given by the spectral zeros which determines the MA part of the resulting ARMA spectra. According to (ii), these are completely arbitrary, when only a finite set of interpolation values are known, such as finitely many covariance samples. Practical rules for selection of such parameters, in absence of prior information about the process, need to be worked out. In cases where spectral zeros of the nominal power spectrum are known *a priori* or can be estimated from longer data records, these same zeros can be enforced to coincide with the spectral zeros of the estimates of the power spectrum, without unduly increasing the complexity of the filters.

Appendix A. Positive real functions and ARMA models

When the power spectrum $\Phi_y(z)$ of the process y is rational, it decomposes into a sum

$$\Phi_y(z) = \Phi_{\text{nondet}}(z) + \Phi_{\text{determ}}(z)$$

of a positive-real function $\Phi_{\text{nondet}}(z)$ with no poles on $|z| = 1$, and a second positive-real term $\Phi_{\text{determ}}(z)$ with (simple) poles on the unit circle and a real part which is zero a.e. on the circle. Then, $\Phi_{\text{nondet}}(z)$ relates to the "non-deterministic part" of y and $\Phi_{\text{determ}}(z)$ to the "deterministic part".

More specifically, let $\Phi_y(z) = \frac{b(z)}{a(z)}$ where a, b are co-prime polynomials, and $\text{Re}\{\Phi_y(e^{i\theta})\} = |\frac{\eta(e^{i\theta})}{a(e^{i\theta})}|^2$, where $\eta(z)$ is a stable polynomial such that

$$\bar{b}(z^{-1})a(z) + \bar{a}(z^{-1})b(z) = \bar{\eta}(z^{-1})\eta(z).$$

Let ϵ_k, $k = 1, \ldots, \kappa$, denote the roots of $a(z)$ on the circle, and ρ_k corresponding residues so that

$$\Phi_y(z) = \Phi_{\text{nondet}}(z) + \sum_{k=1}^{\kappa} \rho_k \frac{z + \epsilon_k}{z - \epsilon_k}.$$

Given that $\Phi_{\text{nondet}}(z)$ has only roots in $|z| < 1$, since all roots on the circle were absorbed in the second term, it is easy to show that $f_y(z)$ is positive-real if and only if $\Phi_{\text{nondet}}(z)$ is positive real and $\rho_k > 0$ for all k. The last condition is equivalent to requiring that the second term, denoted by $f_{\text{determ}}(z)$, is positive-real with zero real part on the boundary (a.e.).

Now, y can be modeled as $y_1 + y_2$, where y_1 is a non-deterministic process generated as the output of an ARMA filter with unit variance white noise input and transfer function $\frac{\eta(z)}{a(z)}$ (with possible common factors between the two polynomials removed), and y_2 is a deterministic process given, for instance, by $y_2(t) = \sum_{k=1}^{\kappa} r_k \epsilon_k^{it}$ with r_k having zero-mean and variance ρ_k. Thus, the deterministic component can be expressed as a sum of sinusoidal components with frequencies dictated by the angle of the poles of $\Phi_y(z)$ which lie on the circle. The common roots between η and a can

also be thought as uncontrollable modes of a Markovian model which, started from suitable random initial condition, generate sinusoidal components at the output.

It is straightforward to show that the corresponding spectral densities of y_1 and y_2 are $\Phi_{\text{nondet}}(z)$ and $\Phi_{\text{determ}}(z)$, respectively.

Appendix B. Determining the gradient

To any $Q \in \mathcal{S}_+$ there is a unique positive real function f satisfying

$$\frac{\Psi(z)}{Q(z)} = f(z) + f^*(z). \tag{B.1}$$

In fact, the left member is positive on the unit circle, and hence it can be split into a sum of an analytic function $f(z)$ and its conjugate $f^*(z)$. Clearly f is positive real.

In this appendix we first prove that if $Q \in \mathcal{S}_+$ is optimal for the problem to minimize Ψ the function f defined by (B.1) is an interpolant. For this, and for later analysis, we need the directional derivative

$$\delta\ \Psi(Q; \delta Q) = \lim_{\epsilon \to 0} \frac{\Psi(Q + \epsilon\delta Q) - \Psi(Q)}{\epsilon},$$

where δQ is a symmetric pseudo-polynomial such that $Q + \epsilon\delta Q \in \mathcal{S}_+$ for sufficiently small $\epsilon > 0$. Performing the differentiation, we have

$$\delta\ \Psi(Q; \delta Q) = \langle \delta Q, w + w^* - \frac{\Psi}{Q} \rangle, \tag{B.2}$$

which, in view of (B.1), yields

$$\delta\ \Psi(Q; \delta Q) = \langle P, w + w^* \rangle - \langle P, f + f^* \rangle. \tag{B.3}$$

Now, suppose Q is the unique minimizing function. Then, $\delta\ \Psi(Q; \delta Q) = 0$ for all directions δQ. In particular, using

$$\delta Q(z) = \tfrac{1}{2}(\alpha + i\beta)G_k(z) + \tfrac{1}{2}(\alpha - i\beta)G_k^*(z),$$

we obtain via (4.3) and (4.4)

$$\delta\ \Psi(Q; \delta Q) = \alpha \text{Re}\{w_k + w_0\} + \beta \text{Im}\{w_k\} - \alpha \text{Re}\{f(z_k) + f(z_0)\} + \beta \text{Im}\{f(z_k)\} = 0,$$

which holds for all real α, β and $k = 0, 1, \ldots, n$. Successively choosing α or β equal to zero and $k = 0, 1, \ldots, n$, we obtain the interpolation constraints for f, as claimed. Moreover, (B.1) implies that $f(z)$ has as spectral zeros the roots of Ψ. The fact that both Ψ and Q are in $\mathcal{S}_+$ insures that f has degree at most n.

Next, to derive an expression (4.17) for the gradient of Ψ, we shall need the following lemma.

Lemma B.1. *Let* $g_1, g_2 \in H(B)$ *be real functions with minimal realizations*

$$\left[\begin{array}{c|c} A & b_1 \\ \hline c & d_1 \end{array}\right] \quad and \quad \left[\begin{array}{c|c} A & b_2 \\ \hline c & d_2 \end{array}\right].$$

Then

$$\langle g_2, g_1 \rangle = b_1' P b_2 + d_1 d_2 \tag{B.4}$$

$$\langle g_2, g_1^* \rangle = d_1 d_2, \tag{B.5}$$

A NEW APPROACH TO SPECTRAL ESTIMATION

25

where P is the unique solution of the Lyapunov equation

$$P = A'PA + c'c,$$

i.e., the observability gramian.

Proof. First note that

$$\langle z^k, z^\ell \rangle = 0 \quad \text{for } k \neq \ell.$$

Therefore, since

$$g(z) := c(zI - A)^{-1}b + d = d + cbz^{-1} + cAbz^{-2} + \dots \quad \text{for } z \geq 1,$$

and $g^*(z) = g(z^{-1})$, (B.5) follows directly by orthogonality. For the same reason,

$$\langle g_2, g_1 \rangle = \langle g_2 - d_2, g_1 - d_1 \rangle + d_1 d_2.$$

But

$$\langle g_2 - d_2, g_1 - d_1 \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} (e^{-i\theta}I - A')^{-1}c'c(e^{i\theta}I - A)^{-1}d\theta,$$

which is precisely the observability gramian P. Hence (B.4) follows. $\square$

Let $\delta q(z) \in H(B)$ be real function with the property that $\delta Q(e^{i\theta}) := \delta q(e^{i\theta}) + \delta q^*(e^{i\theta}) > 0$ for $\theta \in [-\pi, \pi]$, and consider the directional derivative at $Q \in \mathcal{S}_+$ in the direction δQ. From (B.3) we see that

$$
\begin{aligned}
\delta_\Psi(Q; \delta Q) &= \langle \delta q + \delta q^*, w + w^* \rangle - \langle \delta q + \delta q^*, f + f^* \rangle \\
&= 2\langle \delta q, w \rangle + 2\langle \delta q, w^* \rangle - 2\langle \delta q, f \rangle - 2\langle \delta q, f^* \rangle \\
&= 2\langle \delta q, w \rangle + 2\langle \delta q, w^* \rangle - 2\langle \delta q, \hat{f} \rangle - 2\langle \delta q, \hat{f}^* \rangle,
\end{aligned}
$$

where $\hat{f}$ is obtain by projecting f orthogonally onto $H(B)$ and is computed as in Step 4 in Subsection 4.3. Consequently, it follows directly from Lemma B.1 that

$$\delta_\Psi(Q; \delta Q) = \begin{bmatrix} b_w - b_{\hat{f}} \\ d_w - d'_{\hat{f}} \end{bmatrix} \begin{bmatrix} P_2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} b_\delta \\ d_\delta \end{bmatrix},$$

where $w \sim (b_w, d_w)$ and $\delta q \sim (b_\delta, d_\delta)$. But, by definition,

$$\delta_\Psi(Q; \delta Q) = 2\nabla_q{}'_\Psi \begin{bmatrix} b_\delta \\ d_\delta \end{bmatrix},$$

which establishes the expression (4.17) for the gradient $\nabla_q \Psi$, as required.

26   C. I. BYRNES, T. T. GEORGIOU, AND A. LINDQUIST

Appendix C. The central Nevanlinna-Pick interpolant

Following our earlier notation,

$$\{(z_0, w_0), (z_1, w_1), \ldots, (z_n, w_n)\} \tag{C.1}$$

is a set of interpolation data, closed under conjugation, with $z_0 = \infty$ and $|z_j| > 1$ for $j = 1, \ldots, n$, having a positive definite Pick matrix (3.1). Without loss of generality we assume that $w_0 = 1$. (If this is not the case, we scale all values $w_k \sim \frac{w_k}{w_0}$ and after we obtain an expression for the interpolating functions with this new set of data, we scale back by multiplying with w_0.) We let $\tau(z) := \prod_{j=1}^{n}(z - \bar{z}_j^{-1})$, as before, and define

$$\hat{B}(z) := \frac{\tau^*(z)z^n}{\tau(z)}$$

and $\hat{w}(z)$ to be the unique function in $H(\hat{B})$ which interpolates

$$\hat{w}(z_j) = w_j \quad \text{and} \quad j = 1, 2, \ldots, n.$$

Note that $\hat{B}(z)$ differs from our earlier Blasckhe product $B(z)$ by a factor of z, and that $\hat{w}(z)$ is not required to satisfy $\hat{w}(\infty) = w_0$. The reason for this departure from our earlier notation is that we solve the interpolation problem in two steps. We first consider the Nevanlinna-Pick problem with data

$$\{(z_1, w_1), \ldots, (z_n, w_n)\} \tag{C.2}$$

and then modify the solution to account for the interpolation at infinity. The reason will become apparent when we compare the entropy of the various interpolants.

We first follow the steps of the Ball-Helton theory [1], see also [15, Section 8.3] where the Nehari problem is solved in a similar manner. This is an alternative approach to the standard Schur algorithm for constructing a linear fractional transformation that generates all solutions to such an interpolation problem.

Define

$$J := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \hat{G}(z) := \begin{bmatrix} 1 & 0 \\ \hat{w}(z) & \hat{B}(z) \end{bmatrix},$$

and let $\hat{L}(z)\hat{G}_o(z)$ denote a J-inner/outer factorization of $\hat{G}(z)$, i.e.,

$$\hat{G}(z) = \hat{L}(z)\hat{G}_o(z)$$

where $\hat{L}(z)$, $\hat{G}_o(z)$ and $\hat{G}_o(z)^{-1}$ are analytic in $|z| \geq 1$ and

$$\hat{L}^*(z)J\hat{L}(z) = J.$$

It turns out that a function $\hat{f}(z)$ is positive real and satisfies $\hat{f}(z_j) = w_j$ for $j = 1, 2, \ldots, n$ if and only if

$$\hat{f}(z) = \frac{b(z)}{a(z)} \quad \text{and} \quad \begin{bmatrix} a(z) \\ b(z) \end{bmatrix} = \hat{L}(z) \begin{bmatrix} 1 + Y(z) \\ 1 - Y(z) \end{bmatrix}$$

and $Y(z)$ analytic in $|z| \geq 1$ with modulus bounded by one (i.e., bounded real). Briefly, this is due to the fact that the interpolation conditions are satisfied if and only if the "graph" of $\hat{f}(z)$, thought of as a multiplication operator, is in the range of $\hat{G}(z)$. In turn, the range of $\hat{G}(z)$ coincides with the range of $\hat{L}(z)$ since they only differ by a right "outer" factor. Finally, positive-realness amounts to $a(z) + b(z)$ having an

analytic inverse, and the graph symbol $(a(z), b(z))'$ being "positive" with respect to the indefinite J-inner product, i.e., that

$$[a^*(z), \quad b^*(z)] J \begin{bmatrix} a(z) \\ b(z) \end{bmatrix} \geq 0, \quad \text{for } |z| = 1.$$

The latter condition is guaranteed by the fact that

$$[1 + Y^*(z), \quad 1 - Y^*(z)] J \begin{bmatrix} 1 + Y(z) \\ 1 - Y(z) \end{bmatrix} = 2(1 - |Y(z)|^2) \geq 0, \quad \text{for } |z| = 1,$$

and the fact that $\hat{L}(z)$ is J-unitary. In fact, $\frac{1 - Y(z)}{1 + Y(z)}$ is a general representation for a positive real function. The proof that $a(z) + b(z)$ has an analytic inverse is analogous to [15, Section 8.3, Lemma 4].

The J-inner/outer factorization of $\hat{G}(z)$ can be constructed in a variety of ways. One approach is to obtain first a canonical factorization of

$$\begin{bmatrix} 1 & \hat{w}^* \\ 0 & \hat{B}^* \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \hat{w} & \hat{B} \end{bmatrix} = \hat{G}_o^*(z) J \hat{G}_o(z)$$

where $\hat{G}_o(z)$ is outer, i.e., both $\hat{G}_o(z)$ as well as $\hat{G}_o(z)^{-1}$ are analytic in $|z| \geq 1$. To this end, note that

$$\begin{bmatrix} 1 & \hat{w}^* \\ 0 & \hat{B}^* \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \hat{w} & \hat{B} \end{bmatrix} = \begin{bmatrix} \hat{w}^* & 0 \\ \hat{B}^* & 0 \end{bmatrix} + \begin{bmatrix} \hat{w} & \hat{B} \\ 0 & 0 \end{bmatrix},$$

assume a state-space realization

$$\begin{bmatrix} \hat{w} & \hat{B} \\ 0 & 0 \end{bmatrix} = c(zI - a)^{-1}b + d,$$

and consider the Riccati equation

$$P - a'Pa - (c' - a'Pb)(d + d' - b'Pb)^{-1}(c - b'Pa) = 0. \tag{C.3}$$

It turns out that (C.3) has a unique solution P such that with c_+, d_+ defined by

$$\begin{bmatrix} c'_+ \\ d'_+ \end{bmatrix} J [c_+, d_+] = \begin{bmatrix} P - a'Pa & c' - a'Pb \\ c - b'Pa & d + d' - b'Pb \end{bmatrix} =: \mathcal{M}, \tag{C.4}$$

such that the matrix $a - bd_+^{-1}c_+$ has all its eigenvalues in $|z| < 1$. For this choice of c_+, d_+, it follows that

$$\hat{G}_o(z) := c(zI - a)^{-1}b_+ + d_+$$

is the required outer factor. In fact P can be obtained as

$$P = \lim_{k \to \infty} P_k$$

with P_0 equal to the identity matrix I and

$$P_{k+1} = a'P_k a + (c' - a'P_k b)(d + d' - b'P_k b)^{-1}(c - b'P_k a). \tag{C.5}$$

113

Then, c_+, d_+ can be obtained by first obtaining a singular value/vector decomposition

$$\mathcal{M} = U \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & -1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} U' \tag{C.6}$$

and taking

$$\begin{bmatrix} c_+ \\ d_+ \end{bmatrix} \times \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

to be equal to the first two columns of U. Having obtained $\hat{G}_o(z)$ we determine

$$\hat{L}(z) = \begin{bmatrix} 1 & 0 \\ \hat{w} & \hat{B} \end{bmatrix} \hat{G}_o^{-1}(z) = c_L(zI - a_L)^{-1}b_L + d_L$$

via standard algebraic manipulations. It turns out that

$$\left[\begin{array}{c|c} a_L & b_L \\ \hline c_L & d_L \end{array} \right] = \left[\begin{array}{c|c} a - bd_+^{-1}c_+ & bd_+^{-1} \\ \hline c - dd_+^{-1}c_+ & dd_+^{-1} \end{array} \right]. \tag{C.7}$$

We now complete one last step which is used to enforce the interpolation at ∞. We first consider the real number x for which

$$[1, \ -1]\, d_L \begin{bmatrix} 1 + x \\ 1 - x \end{bmatrix} = 0.$$

It can be shown that $|x| < 1$. Next we define

$$R_x := \frac{1}{1 - x^2} \begin{bmatrix} 1 + x & 0 \\ 0 & 1 - x \end{bmatrix}.$$

This is a constant "J-unitary" matrix. Right multiplication of $\hat{L}(z)$ by R_x causes a "J-rotation" such that

$$L(z) := \hat{L}(z)R_x = \begin{bmatrix} \hat{L}_{11} & \hat{L}_{12} \\ \hat{L}_{21} & \hat{L}_{22} \end{bmatrix}$$

satisfies

$$L(\infty) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{C.8}$$

for some $\lambda \neq 0$. Now, consider the class $\quad$ of functions f such that

$$f(z) = \frac{b(z)}{a(z)}, \qquad \begin{bmatrix} a(z) \\ b(z) \end{bmatrix} = L(z) \begin{bmatrix} 1 + Y(z) \\ 1 - Y(z) \end{bmatrix}$$

where $Y(z)$ is analytic and $|Y(z)| < 1$ in $|z| \geq 1$, and $Y(\infty) = 0$. Then, it is easy to see that this is the set of all positive real interpolants of the complete set of data (C.2). To see this note that any $f \in \mathcal{F}$ interpolates (C.2) since its graph is in the range of $L(z)$, which is the same as the range of $\hat{L}(z)$. In view of the normalization (C.8), the restriction $Y(\infty) = 0$ is necessary and sufficient for $f(\infty) = w_0$ (which was assumed to be 1).

Finally, we show that the choice $Y(z) = 0$ gives the interpolant which maximizes the expression for the entropy with $\Psi(z) = 1$.

We need the following fact. If $g(z)$ is outer, i.e. analytic with analytic inverse in $|z| > 1$, then

$$\frac{1}{2\pi} \int_0^{2\pi} \log\left(|g(e^{i\theta})|\right) d\theta = \log\left(|g(\infty)|\right).$$

This is a direct consequence of Szegö's theorem [24, p. 19, and also p. 125]. We now compute, for any $f \in \mathcal{F}$, that

$$
\begin{aligned}
\langle \log\left(f(z) + f^*(z)\right), 1\rangle &= \langle \log \frac{a^*(z)b(z) + b^*(z)a(z)}{a^*(z)a(z)}, 1\rangle \\
&= \langle \log(a^*(z)b(z) + b^*(z)a(z)), 1\rangle - \langle \log a^*(z)a(z), 1\rangle \\
&= \langle \log(1 - Y^*(z)Y(z)), 1\rangle - 2\log\left(|a(\infty)|\right)
\end{aligned}
$$

where in the last step we exploit the fact that $a^*(z)b(z) + b^*(z)a(z) = 1 - Y^*(z)Y(z)$ which follows from $L(z)$ being J-unitary, and the fact that $a(z)$ is outer. Since $a(\infty) = L_{11}(\infty) + L_{12}(\infty)$ is independent of $Y(z)$, the choice of $Y(z)$ which maximizes the expression for the entropy is clearly $Y(z) = 0$.

References

1. J.A. Ball and J.W. Helton, *A Beurling-Lax theorem for the Lie group U(m,n) which contains most classical interpolation theory*, J. Operator Theory, 9: 107-142, 1983.

2. J.P. Burg, *A new analysis technique for time series data*, NATO Advanced Institute on Signal Processing with Emphasis on Underwater Acoustics, 1968, reprinted in **Modern Spectrum Analysis**, D.G. Childers, ed., IEEE Press.

3. C.I. Byrnes and A. Lindquist, *On the geometry of the Kimura-Georgiou parameterization of modelling filter*, Int. J. Contr. (**50**): 2301-2312, 1989.

4. C.I. Byrnes, A. Lindquist and T. McGregor, *Predictability and Unpredictability in Kalman filtering*, IEEE Trans. on Aut. Contr., AC-36: 563-, 1991.

5. C.I. Byrnes, A. Lindquist and Y. Zhou, *On the nonlinear dynamics of fast filtering algorithms*, SIAM J. Control and Optimization, 32(1994), 744-789.

6. C. I. Byrnes, A. Lindquist, S.V. Gusev, and A. S. Matveev, *A complete parametrization of all positive rational extensions of a covariance sequence*, IEEE Transactions on Automatic Control, AC-40: 1841-1857, 1995.

7. C. I. Byrnes and A. Lindquist, *On the Partial Stochastic Realization Problem*, IEEE Transactions on Automatic Control, AC-42: 1049-1070, 1997.

8. C. I. Byrnes, H. J. Landau, and A. Lindquist, *On the well-posedness of the rational covariance extension problem*, in Current and Future Directions in Applied Mathematics, eds. M. Alber, B. Hu, J. Rosenthal, Birkhäuser, pp. 83-106, 1997.

9. C. I. Byrnes and A. Lindquist, *On duality between filtering and control*, in Systems and Control in the Twenty-First Century, C. I. Byrnes, B. N. Datta, D. S. Gilliam and C. F. Martin, editors, Birhäuser, 1997, pp. 101-136.

10. C. I. Byrnes, S.V. Gusev, and A. Lindquist, *A convex optimization approach to the rational covariance extension problem*, to appear.

11. C.I. Byrnes, T.T. Georgiou, and A. Lindquist, *A generalized entropy criterion for Nevanlinna-Pick interpolation: A convex optimization approach to certain problems in systems and control*, submitted for publication, *IEEE Trans. on Aut. Control*, July 1998.

12. C.I. Byrnes, T.T. Georgiou, and A. Lindquist, *Convex optimization and homotopy methods in commutant-lifting theory*, in preparation.

13. Ph. Delsarte, Y. Genin and Y. Kamp, *On the role of the Nevanlinna-Pick problem in circuits and system theory*, Circuit Theory and Applications 9 (1981), 177-187.

14. Ph. Delsarte, Y. Genin, Y. Kamp and P. van Dooren, *Speech modelling and the trigonometric moment problem*, Philips J. Res. **37** (1982), 277-292.

15. B. Francis, **A Course in H_∞ Control Theory**, Springer-Verlag, 1987.

16. C. Foias and A. Frazho, The Commutant Lifting Approach to Interpolation Problems, Birkhäuser, Basel, 1990.

17. T.T. Georgiou, Partial Realization of Covariance Sequences, Ph.D. thesis, CMST, University of Florida, Gainesville 1983.

18. T.T. Georgiou, *A Topological approach to Nevanlinna-Pick Interpolation*, SIAM J. on Math. Analysis, 18(5): 1248-1260, September 1987.

19. T.T. Georgiou, *Realization of power spectra from partial covariance sequences*, IEEE Transactions on Acoustics, Speech, and Signal Processing, ASSP-35: 438-449, 1987.

20. T.T. Georgiou, *The interpolation problem with a degree constraint*, IEEE Trans. on Aut. Control, to appear.

21. U. Grenander and M. Rosenblatt, *Statistical Analysis of Stationary Time Series*, Almqvist & Wiksell, Stockholm, 1956.

22. U. Grenander and G. Szegö, *Toeplitz forms and their applications*, Univ. California Press, 1958.

23. S. Haykin, ed., Nonlinear Methods of Spectral Analysis, Springer Verlag, 1983.

24. H. Helson, *Lectures on Invariant Subspaces*, Academic Press, New York, 1964.

25. R.E. Kalman, *Realization of covariance sequences*, in Toeplitz centennial, editor I. Gohberg, Birkhauser Verlag, pages 331-342, 1982.

26. S. A. Kassam and H. V. Poor, *Robust techniques for signal processing*, Proceedings IEEE 73 (1985), 433–481.

27. M.G. Kendall, A. Stuart, J.K. Ord, Advanced theory of statistics, 4th ed., volume 2, Macmillan, New York, 1983.

28. H. Kimura, *Positive partial realization of covariance sequences*, Modelling, Identification and Robust Control (C. I. Byrnes and A. Lindquist, eds.), North-Holland, 1987, pp. 499–513.

29. B. Sz.-Nagy and C. Foias, Harmonic Analysis of Operators on Hilbert Space, North-Holland, Amsterdam, 1970.

30. R. T. Rockafellar, Convex Analysis, Princeton University Press, 1970.

31. D. Sarason, *Generalized interpolation in H_∞*, Trans. American Math. Soc., 127 (1967), 179–203.

32. L.R. Rabiner and R.W. Schafer, Digital Processing of Speech Signals, Prentice Hall, Englewood Cliffs, N.J., 1978.

33. P. Stoica and R. Moses, Introduction to Spectral Analysis, Prentice-Hall, 1997.

34. J. L. Walsh, Interpolation and Approximation by Rational Functions in the Complex Domain, Amer. Math.Soc. Colloquium Publications, 20, Providence, R. I., 1956.

35. D. C. Youla and M. Saito, *Interpolation with positive-real functions*, J. Franklin Institute 284 (Aug. 1967), 77–108.

What is claimed is:

1. A signal encoder for determining a plurality of filter parameters from an input signal for later reproduction of said signal, said encoder comprising a bank of first order filters, each of said filters being tuned to a preselected frequency, and a covariance estimator connected to the output of said filter bank for estimating covariances from which filter parameters may be calculated for a filter to reproduce said signal.

2. The signal encoder of claim 1 wherein said filter parameters comprise specification of filter poles and filter zeros.

3. The signal encoder of claim 2 wherein said filters comprising said bank of filters are adjustable to permit their being tuned to a desired frequency based on a priori information.

4. The signal encoder of claim 2 wherein said filters comprising said bank of filters are adjustable to permit their being tuned to a desired frequency based on properties of said input signal.

5. The signal encoder of claim 4 wherein said properties are measured frequencies of said input signal.

6. The signal encoder of claim 3 wherein the number of filters comprising said filter bank are adjustable.

7. The signal encoder of claim 6 wherein said filter parameters at least partially define an ARMA filter, and wherein one or more filter zeros are preselected to further define said ARMA filter.

8. The signal encoder of claim 7 wherein said ARMA filter is of lattice-ladder architecture.

9. The signal encoder of claim 1 further comprising a signal synthesizer coupled to said signal encoder.

10. The signal encoder/signal synthesizer of claim 9 wherein said signal synthesizer further comprises a decoder for receiving the covariances from said signal encoder and produce a plurality of filter parameters in response thereto, a parameter transformer coupled to said decoder, and an ARMA filter coupled to said parameter transformer, said ARMA filter being adjustable to effect reproduction of said input signal through processing of a preselected excitation signal.

11. The signal encoder/signal synthesizer of claim 10 wherein said ARMA filter is adjustable in response to said parameter transformer output.

12. The signal encoder/signal synthesizer of claim 11 wherein said excitation signal is preselected.

13. The signal encoder/signal synthesizer of claim 12 wherein said excitation signal is determined by said signal encoder and communicated to said signal synthesizer for excitation of said ARMA filter.

14. The signal encoder/signal synthesizer of claim 13 wherein said ARMA filter includes filter zeros, and wherein said filter zeros are preselected.

15. The signal encoder/signal synthesizer of claim 13 wherein said ARMA filter includes filter zeros, and wherein said filter zeros are specified by a set of MA parameters generated by said signal encoder, said set of MA parameters being adjustable in response to said input signal.

16. The signal encoder of claim 1 further comprising a spectral analyzer coupled to said signal encoder, said spectral analyzer determining the power frequency spectrum of said input signal in response to the output of said signal encoder.

17. The signal encoder/spectral analyzer of claim 16 wherein said spectral analyzer includes a decoder for producing a set of filter parameters, and a spectral plotter for producing a response reflective of the power frequency spectrum of the input signal.

18. A device for verifying the identity of a speaker based on his spoken speech, said device comprising a voice input device for receiving a speaker's voice and processing it for further comparison, a bank of first order filters coupled to said voice input device, each of said filters being tuned to a preselected frequency, a covariance estimator coupled to said filter bank for estimating filter covariances, a decoder coupled to said covariance estimator for producing a plurality of filter parameters, and a comparator for comparing said produced filter parameters with prerecorded speaker input filter parameters and thereby verifying the speaker's identity or not.

19. The device of claim 18 further comprising a memory coupled to said comparator for storing said prerecorded speaker input filter parameters.

20. The device of claim 18 further comprising an input device coupled to said comparator to allow for the contemporaneous input of pre-recorded speaker filter parameters by a user.

21. A Doppler-based speed estimator comprising a pulse-Doppler radar for producing an output of Doppler frequencies, a HREE filter coupled to said radar, and a spectral plotter coupled to said HREE filter for determining the power frequency spectrum of said radar output, said power frequency spectrum thereby specifying the speed of any objects sensed by said radar.

22. A device for estimating the delay between any two signals, said device including a sensing device for producing a time based output reflective of any delay desired to be estimated, a Fourier transformer for converting said time based output to a frequency based output, a HREE filter coupled to said transformer, and a spectral plotter coupled to said HREE filter for determining the power frequency spectrum of said time based signal, said power frequency spectrum thereby specifying said delay.

23. A method for analyzing a signal comprising the steps of passing said signal through a bank of lower order filters, each of said filters being tuned to a preselected frequency, and estimating a plurality of covariances from the output of said filter bank, said covariances being sufficient for calculating a plurality of filter parameters for a HREE filter, said HREE filter thereby being capable of reproducing said signal.

24. The method of claim 23 further comprising the step of calculating the HREE filter parameters from said covariances, and adjusting a HREE filter in accordance with said calculated filter parameters for reproduction of said signal.

25. The method of claim 24 further comprising the step of adjusting said filter parameters based on the input signal.

26. A method of verifying the identity of a speaker based on his spoken speech, said method comprising the steps of receiving a speaker's voice, processing said voice input for further comparison by passing it through a bank of lower order filters, each of said filters being tuned to a preselected frequency, estimating a plurality of filter covariances from said filter outputs, producing a plurality of filter parameters from said filter covariances, and comparing said filter

parameters with prerecorded speaker input filter parameters and thereby verifying the speaker's identity or not.

27. A method of estimating a speed of an object with a Doppler-based radar comprising the steps of producing an output of Doppler frequencies with said Doppler-based radar, passing said frequencies through a HREE filter, and determining the power frequency spectrum of said frequencies to thereby specify the speed of said object.

28. A method for estimating the delay between any two signals, said method comprising the steps of producing a time based output reflective of any delay desired to be estimated, converting said time based output to a frequency based output by taking its Fourier transform, and determining the power frequency spectrum of said frequency based signal to thereby specify said delay.

* * * * *